Continuous Professional Development Certificate in Educational Mentorship and Coaching for Mathematics Teachers (CPD-CEMCMT)

Student Manual

Module 2

3rd Edition

Pedagogical Content Knowledge and Gender in Mathematics Education

February 2020

Module code: PDM1142

UNIVERSITY of RWANDA

vvob education for development Rwanda

REB Rwanda Education Board

Please cite this publication as:

UR-CE (2020) Continuous Professional Development Certificate in Educational Mentoring and Coaching for Mathematics Teachers, Student Manual, Module 2, Kigali.

Designed and printed by **KIGALI SUN Ltd**

Continuous Professional Development Certificate

in

Educational Mentorship and Coaching

for Mathematics Teachers (CPD-CEMCMT)

MODULE TWO

Pedagogical Content Knowledge and Gender in Mathematics Education (PDM1142)

3rd EDITION, FEBRUARY 2020

ABOUT THE AUTHORS

Dr Alphonse UWORWABAYEHO obtained his PhD in Mathematics Education, specializing in the integration of ICT in the teaching and learning of mathematics from the University of Bristol, United Kingdom. He holds a certificate of successful completion with Honours on Mainstreaming Early Childhood Education into Education Sector Planning organized by IIEP-UNESCO/UNICEF/GPE from 28 October to 29 November 2019. His research interest lies in teacher professional development on enhancing active teaching learning. Currently, he is leading the Department of Early Childhood and Primary Education at the University of Rwanda-College of Education (UR-CE). He is a member of African Centre of Excellence for Innovative Teaching and Learning Mathematics and Science (ACEITLMS) at the UR-CE.

Théophile NSENGIMANA is a PhD student in Science Education and Lecturer at UR-CE at the Department of Mathematics, Science and Physical Education. He holds a Master's degree in Education Specialised in Mathematics and Science Teacher Education from Naruto University of Education, Japan. Mr. Nsengimana has wide educational and research experience related to science curriculum implementation and teacher education. He is an Associate Member of African Centre of Excellence for Innovative Teaching and Learning Mathematics and Science (ACEITLMS) at the UR-CE.

Sylvain HABIMANA is an Assistant Lecturer in the Department of Mathematics, Science and Physical Education at the University of Rwanda -College of Education (UR-CE). From 2016, he is pursuing PhD studies at UR-CE with mathematics education as specialization. He holds a Master's degree of Education from Kampala International University (KIU), Uganda. He also holds a Bachelor's degree in Mathematics-Physics-Education (MPE) and a Postgraduate Certificate in Learning and Teaching in Higher Education (PgCLTHE) from UR-CE/ former KIE, Rwanda.

Théoneste HAKIZIMANA holds a Master's degree in Mathematics and a certificate of teaching mathematics in higher learning institutions and secondary schools delivered by People's Friendship University of Russia in 1992. He has 27 years of experience in teaching mathematics in various higher learning institutions. He is currently a lecturer and subject leader of mathematics at the department of Mathematics, Science and Physical Education, College of Education, University of Rwanda.

Dr Védaste MUTARUTINYA (PhD) pursued his studies in the Faculty of Sciences of Friendship People's University of Russia (FPUR) at Moscow, where he obtained a BSc's degree (1994), a Master's degree (1996) in Mathematics and Physics and a PhD (2000) in Mathematics. He is the author of several scientific articles. His fields of research are: Functional Analysis with applications and Mathematics Education. Currently he is a Senior Lecturer of Mathematics at the UR-CE at the Department of Mathematics, Science and Physical Education. He is an Associate Member of the African Centre of Excellence and Innovation in Teaching/Learning Mathematics and Sciences (ACEITLMS) at the UR-CE and a Member of the UR-CE research screening and ethics clearance committee. He holds a Postgraduate Certificate in Learning and Teaching in Higher Education.

Nehemiah BACUMUWENDA holds a Master's degree in Public Health from the University of Rwanda, a Bachelor's Degree in Management from Independent University of Kigali and a Bachelor's Degree in Educational Psychology from Adventist University of Central Africa. He is specialised in the Rwandan education system and has more than 30 years of experience in the education sector. He is currently a curriculum developer at Rwanda Education Board.

Dr Leon MUGABO Rugema (PhD) is an experienced science teacher and science teacher educator. He holds a PhD in Science Education. Currently, he is a lecturer in the Department of Early Childhood and Primary Education, School of Education at UR-CE. He is also Head of Teaching and Learning in the Africa Centre of Excellence for Innovative Teaching and Learning Mathematics and Science (ACEITLMS).

Dr Jean Francois MANIRAHO (PhD) is a lecturer and researcher at UR-CE. Over the last 10 years, Dr Maniraho has been lecturing in different higher learning institutions both inside and outside Rwanda. He is specialized in mathematics education with knowledge of teacher education. Based on his background with a Master of Applied Mathematics, he is also a data analyst, with experience in MATLAB.

Clementine Gafiligi UWAMAHORO holds an MBA in International Business at Amity University of India and a BA in Educational Psychology from Adventist University of Central Africa. She is specialized in the Rwandan education system. She has 19 years of experience in the education sector, especially in curriculum development, education planning, school leadership, teacher training, quality insurance and assessment of Education system. She is currently an education advisor for VVOB.

Stefaan VANDE WALLE is education advisor school leadership and STEM education with VVOB. He holds Master's degrees from the University of Leuven, Belgium (geography), Radboud University Nijmegen, The Netherlands (project planning) and the Open University, UK (online and distance education). He has been working for VVOB since 2008 in Cambodia, South Africa and Rwanda. His areas of specialization include school leadership, teacher education, STEM education, the economics of education and online learning.

ACKNOWLEDGEMENTS

We owe a large debt of thanks to the authors of this guide: Dr Alphonse Uworwabayeho, Théophile Nsengimana, Sylvain Habimana, Théoneste Hakizimana, Dr Védaste Mutarutinya, Dr Mugabo Rugema Leon, Dr Jean Francois Maniraho, Clementine Gafiligi Uwamahoro and Stefaan Vande Walle.

We also like to thank the following TTC Tutors and mathematics subject leaders who have played crucial additional roles in developing and reviewing the course text: Jean Damascene Habimana (TTC Kabarore), Theophile Ngizwenayo (TTC Matimba), Jean Pierre Kuradusenge Rukeribuga (TTC Gacuba), Jean de Dieu Rutagengwa (TTC Zaza), James Bayingana (Nyagatare District), Tharcisse Uwimanimpaye (Nyagatare District), Augustin Gabiro (Kirehe District), Barthelemie Niyitegeka (Kirehe District), Joseline Nyiramatabaro (Rusizi District), Dieudonne Nayituriki (Rusizi District), Jean de Dieu Mugabo (Gastibo District), Francois Mbarushimana (Gastibo District), Simeon Nikuze (Nyabihu District), Vedaste Mufaransa Rushema (Nyabihu District), Solonga Nzahemba (Kayonza District), Theophile Ndagijimana (Kayonza District).

This course would not have been possible without the financial support from the Belgian Government and ELMA Foundation.

Finally, we like to thank the Ministry of Education in Rwanda (MINEDUC), Rwanda Education Board (REB) and the University of Rwanda - College of Education (UR-CE) for their continued support to education in Rwanda in general and to this CPD Certificate Programme on Educational Mentoring and Coaching for mathematics school subject leaders (MSSLs).

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF ACRONYMS

CBC	Competence Based Curriculum
CPD	Continuous Professional Development
CLIL	Content and Language Integrated Learning
CoP	Community of Practice
DDE	District Director of Education
DHT	Deputy Head Teacher
HoD	Head of Department
HT	Head Teacher
ICT	Information and communications technology
KBC	Knowledge Based curriculum
LCP	Learner-centred Pedagogy
NT	New Teacher
OECD	Organisation for Economic Cooperation and Development
PCK	Pedagogical Content Knowledge
PP	Policy Priority
RAWISE	Rwandan Association for Women in Science and Engineering
REB	Rwanda Education Board
SBI	School Based In-service
SBM	School Based Mentor
SEI	Sector Education Inspector
SEN	Special Educational Needs
SSL	School Subject Leader
TDMP	Teacher Development and Management Policy
TTC	Teacher Training College
UR-CE	University of Rwanda – College of Education

MODULE 2: PEDAGOGICAL CONTENT KNOWLEDGE AND GENDER IN MATHEMATICS EDUCATION

Introduction

The first module of this programme focused on coaching, mentoring, induction and communities of practice (CoPs). In this second module we will discuss Pedagogical Content Knowledge (PCK) and inclusiveness in primary mathematics education.

We start this module with a discussion of the competence-based curriculum (CBC) for primary mathematics. Secondly, we introduce the key concepts in this module: Pedagogical Content Knowledge (PCK), Mathematical Proficiency, Mathematical Literacy and Learner-Centred Pedagogy. In the third unit, we will introduce some key aspects of successful mathematics instruction. For each aspect, we briefly introduce the concept and relevant research before we move to concrete classroom-based techniques. Unit 4 and Unit 5 are dedicated to gender and inclusive education and assessment respectively. Finally, there is appendix with a variety of classroom activities, questions and problems that you can use in your lessons and sample lesson plans, arranged per content area in the CBC.

Module learning outcomes

By the end of the module participants should be able to:

- Explain the principles of the Competence Based Curriculum for primary mathematics;
- Understand the concepts of Pedagogical Content Knowledge, mathematical proficiency, mathematical literacy and learner-centred pedagogy;
- Demonstrate understanding of key principles of quality mathematics instruction;
- Successfully mentor fellow teachers in teaching mathematics;
- Apply a variety of techniques and approaches to develop knowledge, understanding, problem solving and reasoning skills and appreciation for mathematics with learners;
- Organize professional development activities for mathematics teachers, including providing effective feedback to peers;
- Address gender stereotypes associated with the teaching of mathematics;
- Select and develop appropriate and inclusive teaching and learning materials for mathematics classrooms;
- Make learning mathematics enjoyable for all learners by integrating daily life situations in mathematics lessons;
- Adapt interventions to meet personal and professional development needs in teaching and learning mathematics;
- Create a culture of on-going reflection and learning for improvement;
- Develop an action plan for improving teaching and learning mathematics in their school;
- Believe that all learners can achieve reasonable levels of mathematics proficiency;
- Appreciate collaboration, teamwork and joined leadership within the school.

UNIT 1: ANALYSIS OF THE MATHEMATICS CBC FOR PRIMARY SCHOOLS

Introduction

Activity 1

In groups of 3, discuss the differences between the Knowledge-based curriculum (KBC) as applied until 2015 and the competence-based curriculum (CBC) currently applied in the Rwandan education system.

Secondly, read the assigned section of unit 1. After reading, discuss with your group members about what you found important in the section and identify any elements that are not clear. Prepare to explain what you read to participants who read another section.

In 2016, REB started the implementation of a competence based curriculum (CBC) in pre/primary and secondary education (REB, 2015). A competence is **the ability to use an appropriate combination of knowledge, skills, attitudes and values to accomplish a task successfully**. In other words, it is the ability to apply learning with confidence in a wide range of situations (REB, 2015). Within the CBC framework, teaching and learning are based on competences rather than focusing only on knowledge. However, this does not mean that knowledge is no longer important, but it means that gaining knowledge is the starting point for developing competences.

Learners work on acquiring one competence at a time in the form of concrete units with specific learning outcomes. REB (2015) states that mathematics equips learners with the competences to enable them to succeed in an era of rapid technological change and socio-economic development. Mastery of basic mathematical ideas and operations (mathematical literacy) should make learners confident in problem-solving in life situations. A high-quality mathematics education therefore provides a foundation for understanding the world, the ability to reason mathematically, an appreciation of the beauty and power of mathematics, and a sense of enjoyment and curiosity about the subject.

Learning Outcomes

By the end of this unit, participants should be able to:

- Explain the structure of the CBC of primary mathematics education;
- Explain the use of different components of the CBC of primary mathematics education;
- Continuously reflect on teaching approaches in line with mathematics competence-based syllabus;
- Plan mathematical learning activities that enhance learners' competences and move beyond transferring knowledge.

Section 1: Competences in the CBC

A competence means the **ability to do something successfully or efficiently**. A competence-based curriculum implies that learning activities are chosen so that learners can acquire and apply the knowledge, skills and attitudes to situations they encounter in everyday life. Competency-based curricula are usually designed around a set of key competences that can be cross-curricular or subject-specific. A competence-based curriculum is less academic and calls for a more practical and skills-based approach with more orientation on the working environment and daily life.

The CBC distinguishes between two categories of competences: basic competences and generic competences. **Basic competences** are key competences that were identified basing on expectations reflected in national policy documents. These competences are built into the learner's profile in each level of education and for all subjects and learning areas. Basic competences have been identified with specific relevance to Rwanda. These are literacy, numeracy, ICT, citizenship and national identity, entrepreneurship and business development, science and technology, and communication in the official languages (REB, 2015).

Generic competences are competences which are transferable and applicable to a range of subjects and situations (REB, 2015). They promote the development of higher order thinking skills. They are generic competences because they apply across subjects.

To guide teachers in sequencing teaching and learning activities, competences have been identified at every level of the CBC from the learner profile to the Key Unit Competences. The **learner profiles** describe the general learning outcomes expected at the end of each phase of education. Teachers are responsible to design lesson plans with instructional objectives linked to the **Key Unit Competences** and leading to all competences above.

The key unit competence is the most important element to pay attention to while designing a lesson plan as it determines the instructional objective(s) of each lesson within the unit.

Figure 1 shows the links between the various competences in the CBC.

- **Broad Competences** are formulated for the end of each learning cycle (at the end of Pre-Primary, Lower Primary, Upper Primary, Secondary 3, and Secondary 6). National Exams assess the achievement of these broad competences according to National Assessment Standards.
- **Key competences** are formulated for the end of each grade. Districts and schools design assessment strategies to ensure learners have achieved the necessary competences and qualify for advancement or need further remediation to meet National Assessment Standards.
- **Key unit competences** are formulated throughout the subject syllabus. The syllabus is divided into units of study to organize learning and encourage teachers to focus on specific content related to learners' daily life and cross cutting issues. Each unit aims to develop basic and generic competences which are evaluated through end unit assessment according to National Assessment Standards.
- **Learning objectives** are specific knowledge, skills, attitudes and values learners should gain within lessons to build progressively the key unit competences. Teachers are responsible to prepare lesson plans based on the subject syllabus.

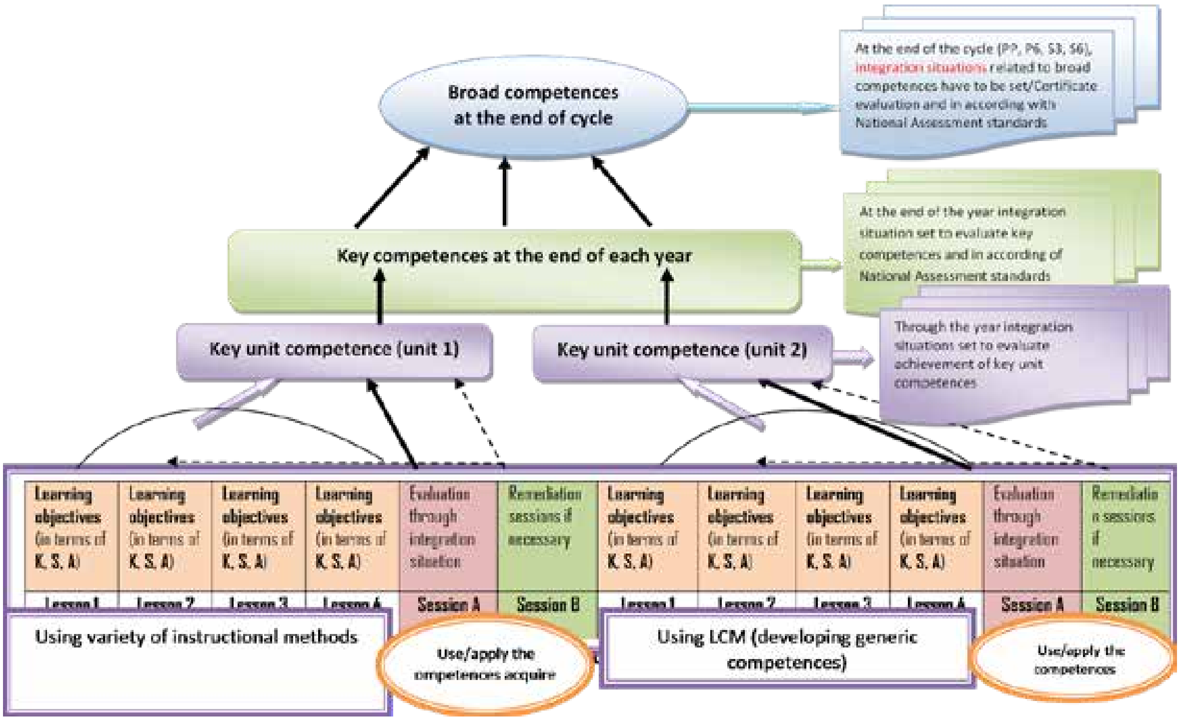


Figure 1: Links between competences elaborated throughout the CBC (REB, 2015)

The key unit competence is the most important element to pay attention to while designing a lesson plan as it determines the instructional objective(s) of each lesson within the unit.

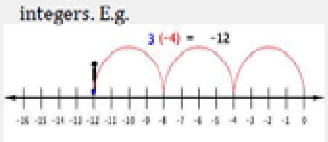
While setting lesson instructional objectives, teachers need to balance Lower Order Thinking Skills (LOTS) and Higher Order Thinking Skills (HOTS). **Higher Order Thinking Skills** (HOTS) are a central element in a competence-based curriculum because they develop the understanding that enables the effective application of knowledge.

Section 2: Mathematics Syllabus

Mathematical concepts are applied in other subjects such as science, technology and in business. Mathematics subject content enhances critical thinking skills and problem solving. Mathematics teaches learners to be systematic, creative and self-confident in using mathematical language and techniques to reason deductively and inductively.

The primary mathematics curriculum is structured into **topic areas, sub- topic areas (where applicable) and in units**. Table 1 shows the structure of each unit.

Table 1: Example of a Unit structure from the mathematics syllabus

Topic Area: NUMBERS AND OPERATIONS				
P.6 MATHEMATICS	Unit 2: Multiplication and division of integers.			No. of Periods: 8
Key Unit Competence: To be able to multiply and divide integers.				
Learning Objectives			**Content**	**Learning Activities**
Knowledge and understanding	**Skills**	**Attitudes and values**		
- Describe the steps taken when multiplying and dividing integers. - Show and explain the concept of integers to solve problems.	- Apply the concepts of multiplication and division to solve problems involving integers. - Carry out multiplication and division of integers. - Explain how integers change in multiplication and division.	- Appreciate the importance of accuracy in multiplication and division of integers. - Respect each other's contribution when working in groups. - Acknowledge the importance of co-operation.	- Multiplication of integers. - Division of integers. - Solving problems involving multiplication and division of integers.	- Learners in their groups do multiplication and division of integers. E.g.  - In groups, learners solve problems involving multiplication and division of integers.
Links to other subjects: *Entrepreneurship: introduction of negative numbers in the context of buying and selling (loss , benefit).*				
Assessment criteria: *Learners should multiply and divide positive integers, negative integers, positive and negative integers.*				
Materials: *Charts should be displayed in class, scissors, markers and masking tape.*				

source: MINEDUC, 2015

Section 3: Lesson Planning

Planning a lesson is an important responsibility for a teacher. A lesson plan is the teacher's guide for running a lesson: It includes the goal (what the students need to learn), how the goal will be reached (methods, procedures) and a way of measuring if the goal was reached (test, activity, homework etc. (REB, 2015). Key elements in developing a lesson plan are summarized in Figure 3.

1. *Check your scheme of work*

At the start of every academic year, teachers develop a Scheme of Work based on the subject syllabus, the school calendar and the time allocated to the subject per week. For lesson plan preparation, consider the following questions:

- What lesson have you planned to teach in a period, such as a term, a month and a week?
- What key competences do you hope to develop by the end of unit?

2. *Identify relevant generic competences and crosscutting issues*

Each lesson must address generic competences and crosscutting issues. In the lesson plan template, there is a section titled 'Competence and crosscutting issues to be addressed'. In this section, you can describe what learners should be able to demonstrate and how the teaching and learning approaches will address these crosscutting issues.

3. *Set instructional objectives*

An instructional objective should have five components. The following steps can guide you to formulate an instructional objective:

a. Reflect on the conditions under which learners will accomplish the assessment task (teaching aids, techniques, outdoors or indoors);

b. Determine who you are talking about (learners);

c. Identify at least one measurable behaviour (knowledge, skills, attitude or values) that you are looking for – evidence of learners' activity. Use a verb which describes the result of learning activities. (e.g. read, write, explain, and discuss). Aim for Higher Order Thinking Skills;

d. Include the content of the activity. You can take this from the subject syllabus.

e. Set standards of performance. Write down the criteria for minimum acceptable performance (for example time, number of correct answers, presence of expected/shared values);

f. Identify the types and number of learners with learning disabilities in the section 'Type of Special Educational Needs and the number of learners in each category, insert the type of disability that you have identified in your class and the number of learners with that disability. In addition, note how these learners will be accommodated in the learning activities.

Education policy targets learners with disabilities who qualify (through standardized SEN assessment) for adjusted educational provisions, or/and who meet barriers within the ordinary education system (REB, 2015). The group includes:

a. Learners with functional difficulties, including physical and motoric challenges, intellectual challenges, visual impairments, hearing impairments, speech impairments;

b. Learners with learning disabilities and learning difficulties (dyslexia, dyscalculia…);

c. Learners with social, emotional and behavioural difficulties (Attention Deficit Hyperactivity Disorder, Asperger's Syndrome…);

d. Learners with curricula-related challenges and difficulties to comprehend or use the teaching languages (including linguistic minorities);

e. Learners with health challenges.

4. *Identify organizational issues*

This part of the lesson plan as about creating positive learning environments, specifically related to physical safety and inclusion. In the section titled “Plan for this class (location: in / outside)”, you can write down where you will hold the lesson, seating arrangements etc.

5. *Decide on teaching and learning activities*

In this part, the teacher summarizes the learning and teaching process including main techniques and resources required. In the column “teacher’s activities”, you describe the activities using action verbs. Questions and instructions from the teacher are also written in this column. In the column “learner activities”, the teacher describes the learner activities, findings and answers. Activities or answers which don’t fit in the column, can be added in an appendix. The teacher specifies whether activities are carried out individually, in small groups or with the whole class.

In the column of steps and timing in the lesson plan format, there are three main steps: introduction, development of the lesson and conclusion.

- **Introduction** is where the teacher connects the lesson with the previous lesson. For example, the teacher organizes a short discussion to encourage learners to think about the previous learning and connect it with the current instructional objective.

- **Development** of the lesson. Depending on the lesson, the development of the content will go through the following steps: discovering activities, presentation of learners’ findings, exploitation and synthesis/summary.

- In **conclusion**, the teacher assesses the achievement of instructional objectives and guides learners to make the connection to real life situations. You may end with homework.

6. *Decide on the timing for each step*

You need to allocate time for each step of the lesson. It is advised to reserve time for learners to write down key words or a summary of the content in their notebooks.

The lesson plan has two main parts: a basic information part (Figure 2) and a specific part (Figure 3).

a) Basic information part components

School Name: ………………**Student-Teacher's name:** …………**Reg.No and Combination:**………………

Term	**Date**	**Subject**	**Form or Class**	**Unit Nº**	**Lesson Nº**	**Duration**	**Class size**
	……/……/20……	…..	….	….	… of ….	…	….
Type of Special Educational Needs to be catered for in this lesson and number of learners in each category							
Unit title							
Key Unit Competence							
Title of the lesson							
Instructional or learning Objective	Knowledge and Understanding: - - - Skills: - - Attitudes and Values - -						
Plan for this Class (location: in / outside)							
Learning Materials (for all learners)							
References(at least 3 references including subject curriculum)	1. References should be written here in this order; That is Author, Year of publication 'in bracket', Title of the reference 'underlined', and consulted pages of the reference. Example: Kamba, G. (2010). <u>Secondary Mathematics (Student's Book III). MK PublishersLtd</u>. Page …						

Figure 2: Basic information part of the CBC lesson plan (REB, 2015)

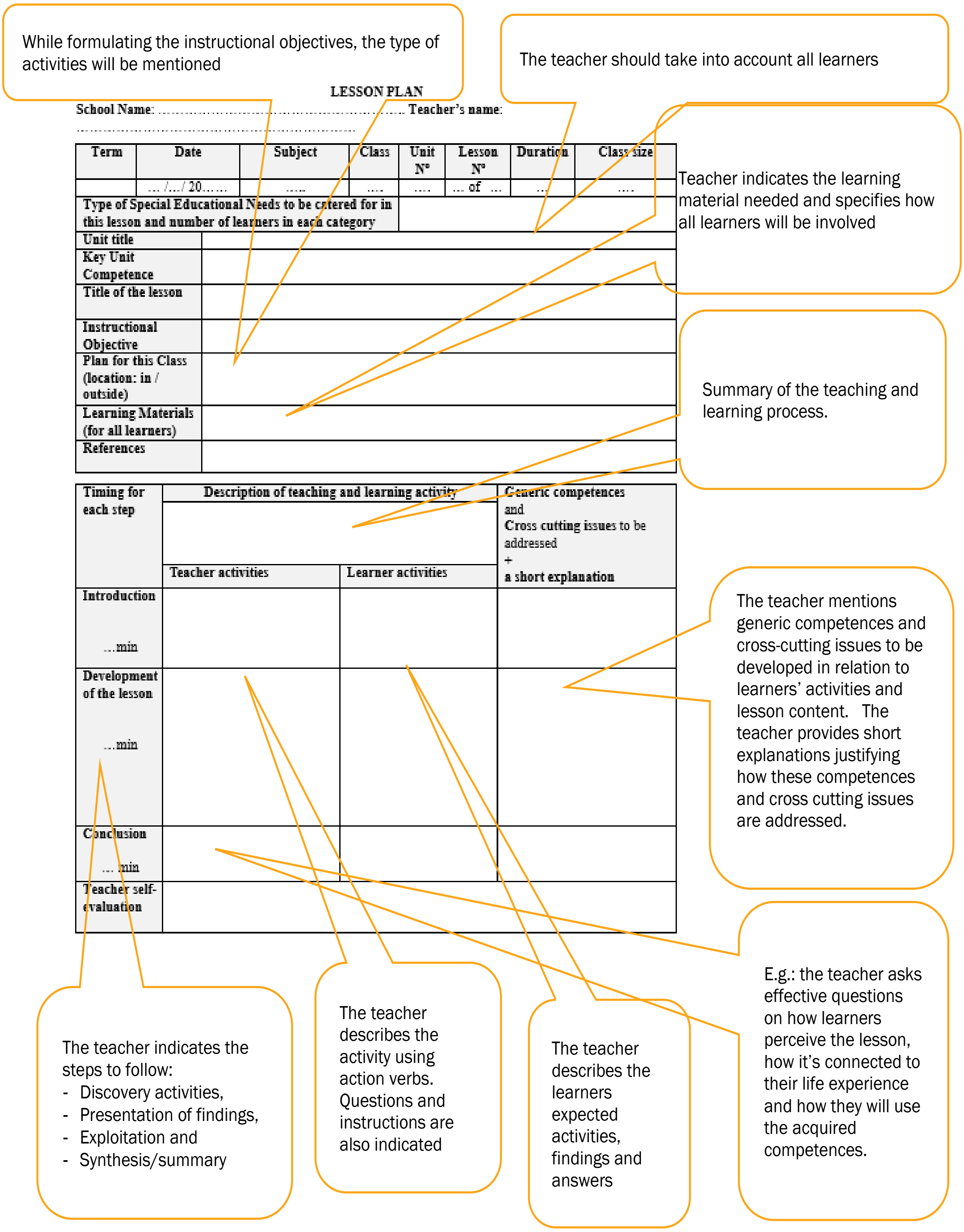


Figure 3: Specific Part of the CBC Lesson Plan (REB, 2015)

UNIT 2: KEY CONCEPTS IN MATHEMATICS EDUCATION

Introduction

This unit discusses several key concepts on teaching mathematics that form the basis for this Programme: Pedagogical Content Knowledge (PCK) for mathematics, Mathematical Proficiency, Mathematical Literacy and Learner-Centred Pedagogy. Familiarity with these concepts will enable you to improve your teaching and your support to your fellow teachers.

Activity 2

Read the case stories in the boxes below. List good points and points for improvement.

Case story 1: Claire's Lesson on introducing ratios (Chick & Harris, 2007)

Claire began by showing the students a 2cm×3cm rectangle and reviewing the definition of area and perimeter, highlighting meaning and units. She then demonstrated how to colour the rectangle in such a way that for every square that was coloured in red, two squares had to be coloured in blue, before having a student repeat this process for another 2cm×3cm rectangle. She asked students how many squares were coloured red and how many were blue, but before this had been answered one student pointed out that 1/3 of the rectangle was red and 2/3 was blue.

This unexpected response allowed Claire to explore the connection between fractions and the situation that they had, highlighting that the 1/3 came from the fact that 2/6 of the squares in the rectangle had been coloured red. After showing students that they could also colour half squares while still achieving one red colouring for every two blue, and demonstrating such an example, she asked students to find the different colourings of the 2cm×3cm rectangle using the "one red for every two blue" scheme. As they started work, she drew from them the need to work systematically, suggesting that they start with whole square colourings first, and attend to the different possible positions of the red coloured squares.

After allowing students to explore the problem for about 15 minutes she had students talk about how they had worked through all the possible arrangements of red and blue colourings, incorporating some discussion about how equivalent arrangements can arise by “flipping” (reflecting) arrangements already found. She concluded her use of this example by emphasising to students that although they had produced many different arrangements, the area of red in all cases was 2cm2 and the area of blue was 4cm 2.

Claire did not mention ratio at all during the first 25 minutes of the lesson; the emphasis seemed to be on area, working systematically, and then, briefly, ideas of symmetry.

However, her choice of the 2cm×3cm rectangle, and the simple proportion “one red to two blue” allowed students to consider the area, problem solving, and symmetry ideas—with fractions receiving some consideration as well—while building a foundation for talking about ratio. It was only after this exploration of a single example that she defined ratio, using the 2 red to 4 blue idea, helping students to see the connection to fractions, identifying the connection between the parts and the whole, getting students to simplify the ratio 2:4 to the “basic” ratio 1:2, and linking this back to her original colouring instruction to colour 1 red and 2 blue. The example used—the 2cm×3cm rectangle and the ratio 1:2—was used for teacher demonstration with a conceptual focus, but was also used as a student task, and the focus was on conceptual ideas rather than procedural ones.

Case story 2: Jean's Lesson on introducing ratios (Chick & Harris, 2007)

Jean began his lesson by asking the class if anyone knew what a ratio was, with the students' responses suggesting that some had heard the term but had little idea about its meaning. One student used the expression "a ratio of one to two" but could not illustrate its meaning. Jean then explained that ratio is associated with fractions or proportion and is used to show the amounts that comprise a whole. His explanation was, at this stage, imprecise and given without an illustrative example. He then invited ten students to stand at the front of the class, highlighted that the ten was the whole, and asked students to determine what proportion of boys and what share were girls. He showed students how to write this as 3:7 and emphasised that 3+7 gives ten, the total in the group. He had students rearrange themselves to show the ratio of their favourite colour (blue or red), which turned out to be 5:5.

Based on these examples, Jean then gave the students some notes about ratio. He used three different examples based in the same context: a discrete collection that he divided into two groups in two different ways. He used the examples to demonstrate the notation of ratio and how to say it, and to remind students about the whole and that ratio compares two numbers. His emphasis during the introductory exposition was partly conceptual, but with a strongly procedural emphasis in the discussion of the way to write and say ratios. His notes on the board for students to copy included an extra example, 1:5, which initially had no physical context, and which his notes suggested could also be written as 1/5. This was done without comment or additional explanation.

He later illustrated the 1:5 example in the context of making lemonade, where he highlighted that one part of fruit juice and five parts of water should be used, to give a total of six equal parts. He also clarified that the actual size or amount of these equal parts did not matter, provided all parts are equal, and emphasised that the order of the numbers in the ratio matters.

Learning Outcomes

By the end of this unit, participants should be able to:

- Demonstrate understanding of the interrelation between procedural fluency, conceptual understanding, strategic competences, adaptive reasoning and productive disposition for achieving mathematical proficiency and literacy;
- Explain the importance of learning mathematics at primary education level through learners' daily life experiences;
- Support fellow teachers to use mathematics to solve problems related to learners' daily experiences;
- Respect the diversity in feelings, opinions and prior knowledge in learners.

Section 1: Pedagogical Content Knowledge for Mathematics

Activity 3

Think about an excellent mathematics teacher that you have encountered. What made that teacher excellent?

Write down your ideas and discuss them with your neighbour.

A mathematics teacher does not simply teach mathematics but helps learners to learn mathematics (Figure 4).

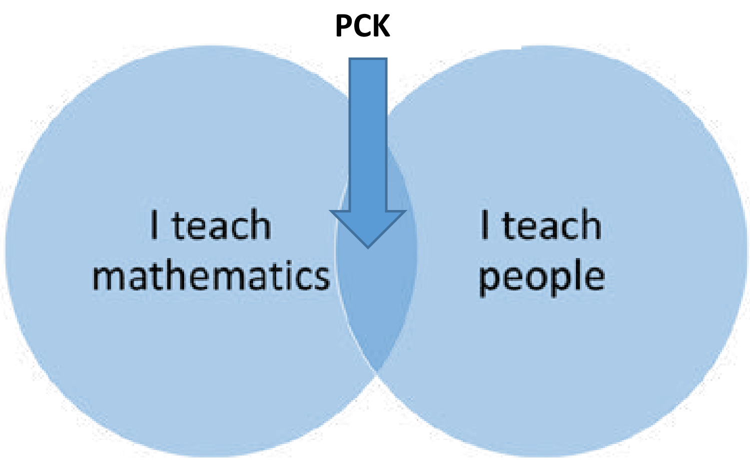


Figure 4: PCK for maths at the intersection of teaching maths and teaching people (VVOB)

Shulman (1986) identified **three types of knowledge** that teachers need to teach well: **content knowledge, curriculum knowledge and pedagogical content knowledge**.

For example, a teacher who plans to teach a lesson on multiplying decimals needs to know a lot more than how to do the multiplication (Ball, 1990, p. 448):

> *"The teacher had to know more than how to multiply decimals correctly herself. She had to understand why the algorithm for multiplying decimals works and what might be confusing about it for students. She had to understand multiplication as repeated addition and as area, and she had to know representations for multiplication. She had to be familiar with base-ten blocks and know how to use them to make such ideas more visible to her students. Place value and the meaning of the places in a number were at play here as well. She needed to see the connections between multiplication of whole numbers and multiplication of decimals in ways that enabled her to help her students make this extension. She also needed to recognize where the children's knowledge of multiplication of whole numbers might interfere with or confuse important aspects of multiplication of decimals. And she needed to clearly understand and articulate why the rule for placing the decimal point in the answer – that one counts the number of decimals places in the numbers being multiplied and counts over that number of places from the right – works. In addition, she needed an understanding of linear and area measurement and how they could be used to model multiplication. She even needed to anticipate that a fourth-grade student might ask why one does not do this magic when adding or subtracting decimals and to have in mind what she might say."*

"What you do when you're teaching is you think about other people's thinking. You don't think about your own thinking; you think what other people think. That's really hard." -Deborah Ball

For example, many people will be able to solve the multiplication below. The teacher's task is not only to calculate the correct answer, but to recognize **why learners make certain mistakes** and **adapt teaching accordingly**.

Multiply:
```
   49
 x 25
 ----
```

Consider the following incorrect answers from learners. How was each answer produced? What misunderstandings might lead a student to make these errors? This specialized knowledge is less likely to be present with people who are good at doing mathematics, but don't have any teaching experience. Recognizing the underlying thoughts from learners that cause these errors is a crucial skill for teachers. A teacher who can only say: "Your answer is wrong", is not more helpful than a doctor who says that you're sick but can't make a good diagnosis.

How was each answer produced?
What might lead a student to make these errors?

```
a.    49        b.    49        c.    49
    x 25            x 25            x 25
    ----            ----            ----
     405             225            1250
    108              100              25
    ----            ----            ----
    1485             328            1275
```

Discussion: what misunderstandings could lead to each of these three answers?

a. 1485

- What mathematical steps are involved? Multiply 9 x 5, which produces 45. Write down the 5 and carry the 4. Add the 4 to the other 4 in the tens column, which yields 8, and multiply 8 x 5, which is 40. Write down 40. Next, multiply 9 x 2, which equals 18. Write down 8 and carry the 1; as before, add the 1 to the 4 before multiplying, i.e., 5 x 2, which equals 10.

- What is the main issue to understand? This process adds the carried ten in before multiplying, instead of afterwards.

b. 325

- What mathematical steps are involved? Multiply 25 x 9 first (bottom up). This yields 225. Then multiply 25 x 4, which equals 100.
- What is the main issue to understand? This process starts with the bottom number instead of with the top as is conventional. This is mathematically valid because multiplication is commutative and so the order in which one multiplies does not matter. However, 25 x 4 is really 25 x 40, which would produce 1000.

c. 1275

- What mathematical steps are involved? Round 49 up to 50, then multiply 50 x 25, which is 1250. Then add 25 to 1250 because 49 is less than 50.
- What is the main issue to understand? This process compensates in the wrong direction — i.e., adds 25 to the 1250 instead of subtracting. Someone might do this because with the conventional procedure one adds together the two separate answers.

The knowledge that teachers need for teaching maths goes **beyond the mathematics content knowledge** and includes:

- Guiding instruction starting from learners' prior knowledge and development level
- Selecting appropriate examples in the right order
- Using correct mathematical language and notation
- Using learners' errors as rich sources of information
- Anticipating and reacting immediately to learners' responses ("learning to see more in the moment")
- Selecting appropriate routine and non-routine word problems
- Asking learners questions that guide them in their learning process, mainly using open questions

- Choosing, using and connecting different representations of a concept
- Linking mathematics with daily life applications

Activity 4

Referring to the case stories of Claire and Jean, do both teachers demonstrate strong PCK in their lessons? Give reasons to support your position.

Discussion of case stories

One of the best things about Claire's lessons was her capacity to make connections among a range of mathematical topics. She made strong use of area understanding (and referred to perimeter in passing), was careful in establishing the links to fractions, and used correct terminology such as "factor". Claire's work with the rectangles and letting students build up an equivalent ratio from the simpler one and allowing students to see multiple configurations of square colourings all of which show 2:4, have made it much easier for them to understand the idea of simplifying the ratio.

Jean certainly appeared to understand the content but was not explicit about the connection between a ratio and its simplified form. Although he recognised that fractions and ratios are linked, he did not address the connection between the ratio 1:5 and the fraction 1/5.

The situations highlight the difficulty of selecting appropriate examples and using them effectively to illustrate general principles. The need to choose suitable representations is particularly important. The discrete representations of ratios apparent in the groups of people used in Jean´s class restrict full understanding of ratio, when compared to the continuous area model used in Claire's class, especially as she allowed students to colour half squares and to consider quarters as well. The sequencing of examples was also important, with Claire building up non-simplified ratios before considering simplification and equivalence, and she also tried to ensure that students were prepared for the problems on the worksheet.

Strengthening PCK is a key instrument to improve the quality of teaching and learning. PCK develops with teaching experience. However, it doesn't come automatically, but requires **continuous professional development and reflection**.

You can strengthen your PCK as a teacher by doing the following:

- Figure out why procedures work, not just how to do them;
- Try to solve problems in more than one way;
- Listen to and ask questions to learners about their work, especially when they are struggling;
- Study learners' thinking and work;
- After a lesson, reflect critically on what went well and what could be improved, preferably with a colleague;
- Prepare lessons together with peers, observe lessons from your colleagues and discuss them afterwards.
- Invite colleagues to observe your lessons and give you feedback.

Section 2: Mathematical Proficiency

Mathematics teachers should not only focus on making sure that learners can do the necessary procedures. Equally important aspects of teaching maths are to help them see the relations between concepts and to motivate them to learn mathematics. The National Council of Teachers of Mathematics (NCTM) in the US has developed the concept of **mathematical proficiency** (National Research Council, 2002).

According to the Council, mathematical proficiency has **five components** (Figure 5):

1. conceptual understanding: comprehension of mathematical concepts.
2. procedural fluency: skill in carrying out procedures flexibly, accurately, efficiently, and appropriately.
3. strategic competence: ability to formulate, represent and solve mathematical problems.
4. adaptive reasoning: capacity for logical thought, reflection, explanation and justification.
5. productive disposition (motivation): attitude to see mathematics as sensible, useful and worthwhile, coupled with a belief in diligence and one's own efficacy.

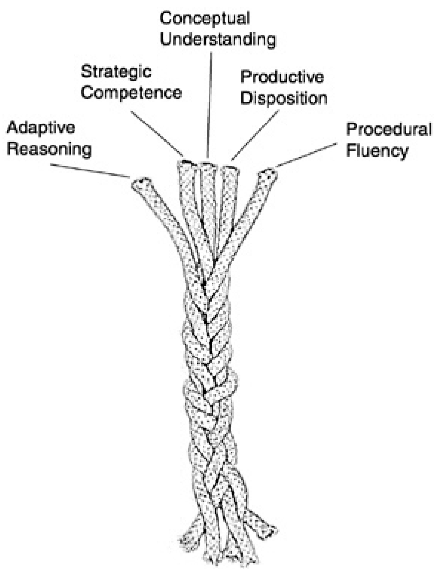


Figure 5: Components of Mathematical Proficiency (National Research Council, 2002)

These five components are related and important for competence-based mathematics teaching. Each component strengthens the others to make learners proficient in mathematics.

For example, procedural fluency and conceptual understanding strengthen each other. As a learner achieves conceptual understanding, he/she will remember procedures better as well. In turn, as a procedure becomes more automatic, the learner can start to think about other aspects of a problem and tackle new kinds of problems, which leads to new understanding.

Let's have a closer look at each component:

1. *Conceptual Understanding*

Students with conceptual understanding know more than isolated facts and methods. They can learn new ideas by connecting them to what they already know.

2. *Procedural Fluency*

Procedural fluency is very important. In daily life, you need to be able to solve certain problems such as additions and multiplications quickly without thinking through the underlying concepts or using a calculator. Also, learners need **basic fluency with procedures** when solving more complicated problems (Burns, 2015).

3. *Strategic Competence*

Strategic competence includes problem formulation and problem solving. Problem solving is more than giving learners problems to solve. Outside of school a big part of the difficulty of problem solving is to figure out what the problem is and formulate the problem in such a way that a learner can use mathematics to solve it.

4. *Adaptive Reasoning*

Adaptive reasoning is being to think logically about the relationships between concepts and situations. It includes estimating the result of a mathematical problem and identifying unrealistic answers. It also means being able to justify one's work with correct mathematical language. You can develop adaptive reasoning with your learners by **giving learners regular opportunities to talk about the concepts and procedures they are using** and let them explain what they are doing and why. Here are some ***questions that develops learners' reasoning skills.***

Example 1

If 49 + 83 = 132 is true, which of the following is true (without calculating) and explain why.

a. 49 = 83 + 132

b. 49 + 132 = 83

c. 132 – 49 = 83

d. 83 – 132 = 49

Research in the US found out that only 61% of American 13-year-olds chose the right answer on this question, which is lower than the percentage of students who could correctly compute the result (National Research Council, 2002).

Example 2

Without calculating, <u>estimate</u> which number is closest to this sum:

12/13 + 7/8

a. 1

b. 2

c. 19

d. 21

Fifty-five percent of American 13-year-olds chose either 19 or 21 as the correct response.

Even small levels of reasoning skills should have prevented this error. Simply observing that 12/13 and 7/8 are numbers less than one and that the sum of two numbers less than one must always be less than two would have made it clear that 19 and 21 were unrealistic answers (National Research Council, 2002).

5. *Productive Disposition*

Productive disposition means seeing mathematics as useful and worthwhile to study, believing that effort in learning mathematics will be rewarded and seeing oneself as an effective learner and doer of mathematics. Productive disposition develops together with the other components and helps each of them develop. For example, as learners build strategic competence in solving problems, their attitudes and beliefs about themselves as mathematics learners become more positive. It is important that students regularly have **success experiences that strengthen their confidence as mathematics learners**. Integrating mathematical games and situations from real life show learners that mathematics can be fun and relevant for their lives.

Box: Further reading

https://buildingmathematicians.wordpress.com/2016/07/31/focus-on-relational-understanding/

http://www.nixthetricks.com/NixTheTricks2.pdf

(pdf in maths resources)

Activity 5

Review the activities per content area in the appendix. Can you find examples of activities that you can use to strengthen each component of mathematical proficiency? Explain your choices.

Section 3: Mathematical Literacy

Activity 6

What do you understand by mathematical literacy?

What have you learnt so far which can help you to develop mathematical literacy with your learners?

The Organisation for Economic Cooperation and Development (OECD) defines mathematical literacy as *a learner's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgements and to use and engage with mathematics in ways that meet the needs of that person's life as a constructive, concerned and reflective citizen* (OECD, 2006). Mathematical literacy is therefore the **ability to use mathematics to solve real-world problems or use mathematics in daily life situations,** such as calculating how much you need to pay in the market.

Activity 7

Can you give examples of how you develop mathematical literacy with your learners?

Mathematical literacy is a key learning outcome for all students, alongside literacy. The term "numeracy' is used as well, which refers to having basic competences in numbers and operations.

Some students struggle to apply knowledge and skills in real life situations, as mathematics requires abstract thinking which can be a difficult transition. Many students also find it challenging to interpret word problems—figuring out exactly what the problem is and identify the steps to find the answer.

Mathematics is everywhere, and it is used in everyday life from cooking, sports, home construction, agriculture, nursing and driving (Figure 6 and Figure 7).

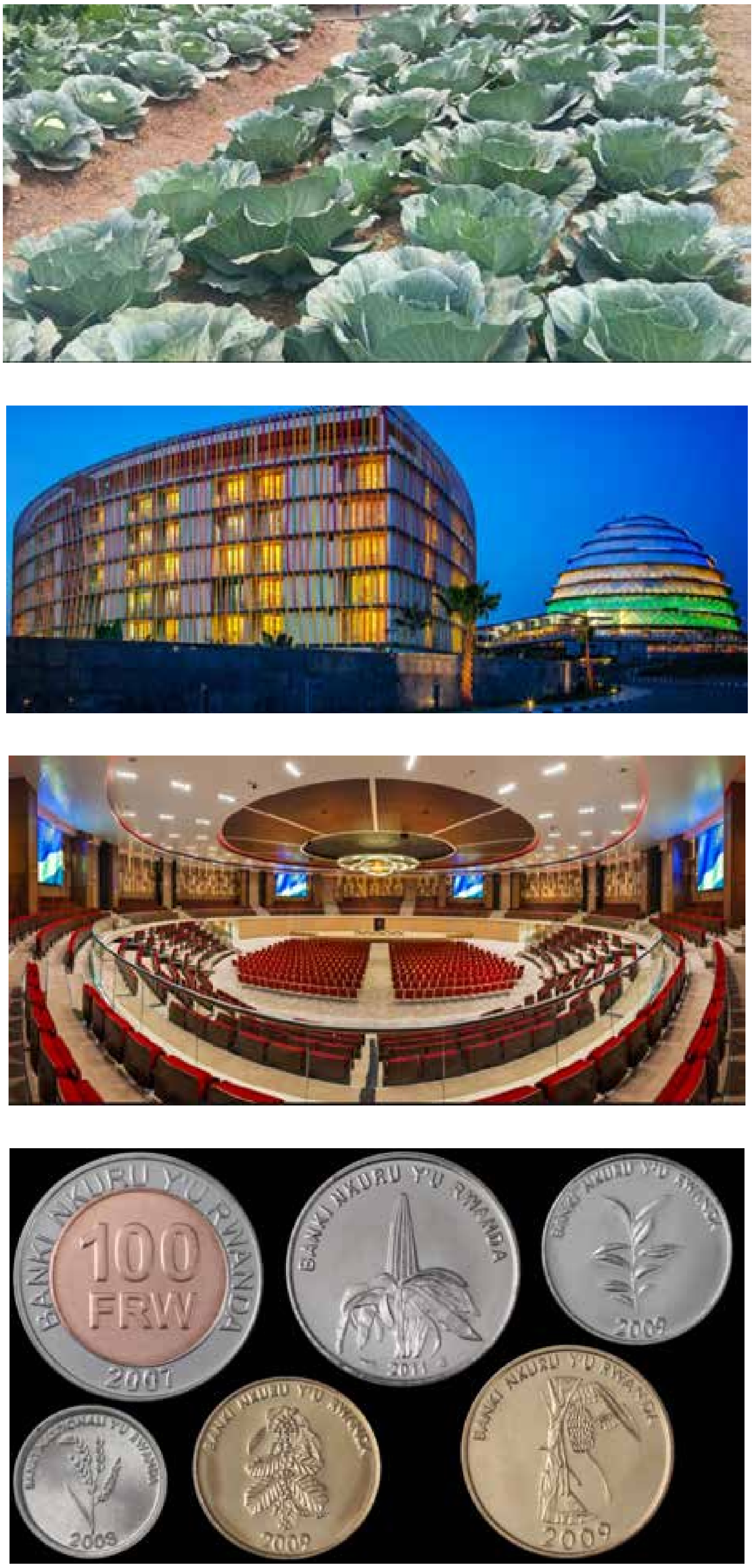


Figure 6: Examples of mathematical concepts in daily life

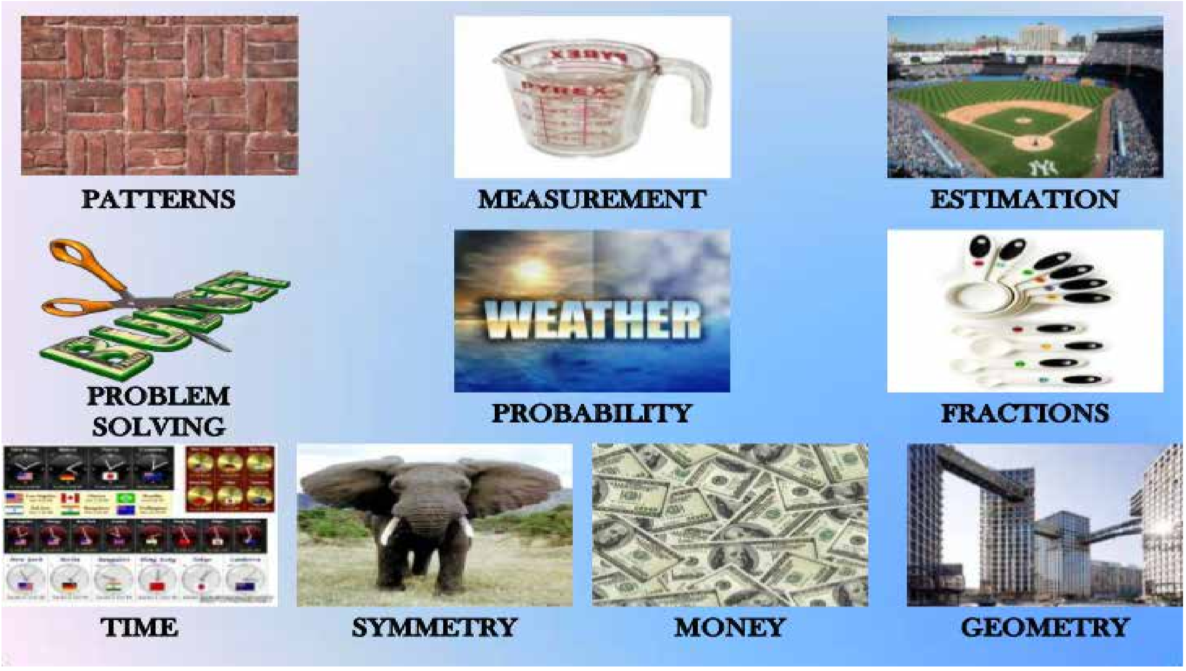


Figure 7: Applications of mathematics in daily life

Source: Jovsan Fernandes

Activity 8

List some situations outside of school for which you have asked your learners to use mathematics during the past month.

As a mathematics teacher, you can play an important role by making meaningful connections between mathematics, the real world and other subjects. This will help learners to realize that mathematics is a way to describe the real world.

Section 4: Learner-Centred Pedagogy (LCP)

What is Learner-Centred Pedagogy?

Activity 9

What do you understand under a learner-centred pedagogy? Can you give examples of LCP from your teaching? Justify why are they examples of LCP.

In learner-centred classrooms, **learners co-influence the teaching and learning process**, in contrast to a teacher-led classroom whereby the teacher is fully in charge of the content, the teaching and learning process.

There is a misconception that learner-centred pedagogy always means working in groups (Nsengimana et al., 2017). Learner-centred pedagogy includes a variety of techniques and approaches, including, but not limited to group work. In unit 3, we discuss some techniques such as open-ended questioning, games, mathematics conversations and problem-solving activities.

UNIT 3: KEY ASPECTS OF MATHEMATICS INSTRUCTION

Introduction

Based on the main ideas that underlie quality teaching of mathematics (Unit 2), we discuss in this unit some key aspects of mathematics teaching that put LCP into practice. We have divided this unit into **7 sections**. In each section, we introduce one aspect of quality mathematics instruction. In each section, we start with the basic principles, followed by concrete techniques.

Learning Outcomes

By the end of this unit, you should be able to:

- Demonstrate understanding of approaches to build mathematical proficiency with learners;
- Apply appropriate methods for teaching and learning mathematics;
- Support fellow teachers with a focus on learner-centred techniques for mathematics teaching;
- Respect of feelings, opinions, people diversity and initiatives of other
- Value social justice and sustainability;
- Appreciate the need for lifelong learning;
- Value collaboration, teamwork and joined leadership within the school.

Activity 10

Individually, complete the self-evaluation that you find in Appendix 2. After completing the self-evaluation, identify for yourself at least three elements of your teaching that you want to improve upon.

Activity 11

Read the case study and discuss the questions.

The goal of this P6 lesson is to calculate the percentage of profit and loss. The teacher starts his lesson by asking the students: What is the definition of profit? What is the definition of loss? *A student gives the answer.* Then the teacher divides the class in 6 groups: *the students who are sitting together at the same table have to arrange themselves in groups.*

The teacher gives each group 5 tomatoes.

After that, he writes down the following questions on the board.

1. if a tomato costs 100 F, calculate the cost of all tomatoes
2. If tomatoes are sold at 150 F each, will the shopkeeper make a profit or a loss?
3. Calculate the profit.
4. How can you find the percentage of profit and loss?

The teacher asks the students to come to the blackboard and write the answer. *Every student who came to the blackboard gets a "clap/ well done", also the students who wrote a wrong answer. For answer 2) a student writes "profit".* The teacher does not accept the answer.

None of the students can find the answer for question 4. The teacher has to help. Next, the teacher gives a new exercise:

- Someone bought a book of 8000 F and sold it for 10 000 F. Calculate the percentage of profit.
- The selling price of a set of books is 500 000 F, but the books were bought for 400.000 F. Calculate the percentage of loss.

Students cannot find the right answer.

In your groups, discuss following questions:

- Is this case study realistic?
- What does the teacher do when students cannot give the right answer? What would you do?
- How does the teacher introduce the new content of the lesson?
- How does the teacher divide learners into groups? How would you do it?
- Did you find working in groups useful in this context? Why (not)?
- What do you think about the choice of examples by the teacher? Did the examples cover all situations?
- What do you think about the use of concrete materials by the teacher? How could you improve it?

Section 1: Questioning

Introduction

Questioning is a key skill for teachers. During an average lesson, teachers ask tens of questions (Lemov, 2015). But what makes a question effective? And how can you use questioning to stimulate thinking, collaboration and motivation with your learners in your mathematics lessons? This section enables you to reflect on and practice effective questioning techniques.

Activity 12

Why do you use questioning in your lessons? Think of as many reasons as possible.

Some **reasons** for teachers **to ask questions** are:

- To check whether students remember what they just learned.
- To recall prior knowledge so they can understand the content of the lesson.
- To 'wake up' students.
- To engage them with new knowledge.
- To focus their attention on what is important.
- To encourage thinking and exploration.
- To let them develop new ideas.
- To connect their old knowledge with new knowledge.

The first three reasons are about **checking learners´ understanding** while the other reasons focus on how **to get and keep learners thinking**. Research has shown that teachers ask many questions to check understanding, whereas they ask relatively few questions to get learners thinking (Wiliam, 2016).

Activity 13

Think in small groups about what a good question means. Make a concept map with criteria of a good question. Put your concept map on the wall, look at the maps from the other groups and discuss areas of agreement and disagreement.

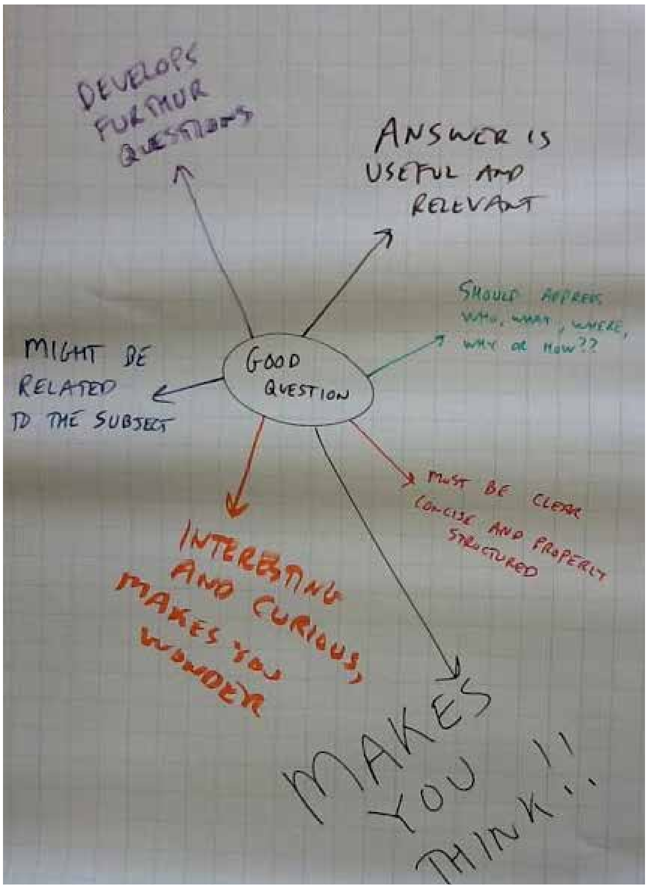


Figure 8: What makes a good question? Example of a Concept map (VVOB, 2017)

Questions are important in a lesson for different reasons. They enable you to involve learners, focus attention on what is important, encourage thinking and exploration and let them develop new ideas, connecting old with new knowledge (Figure 9) (Lemov, 2015; Martino & Maher, 1999).

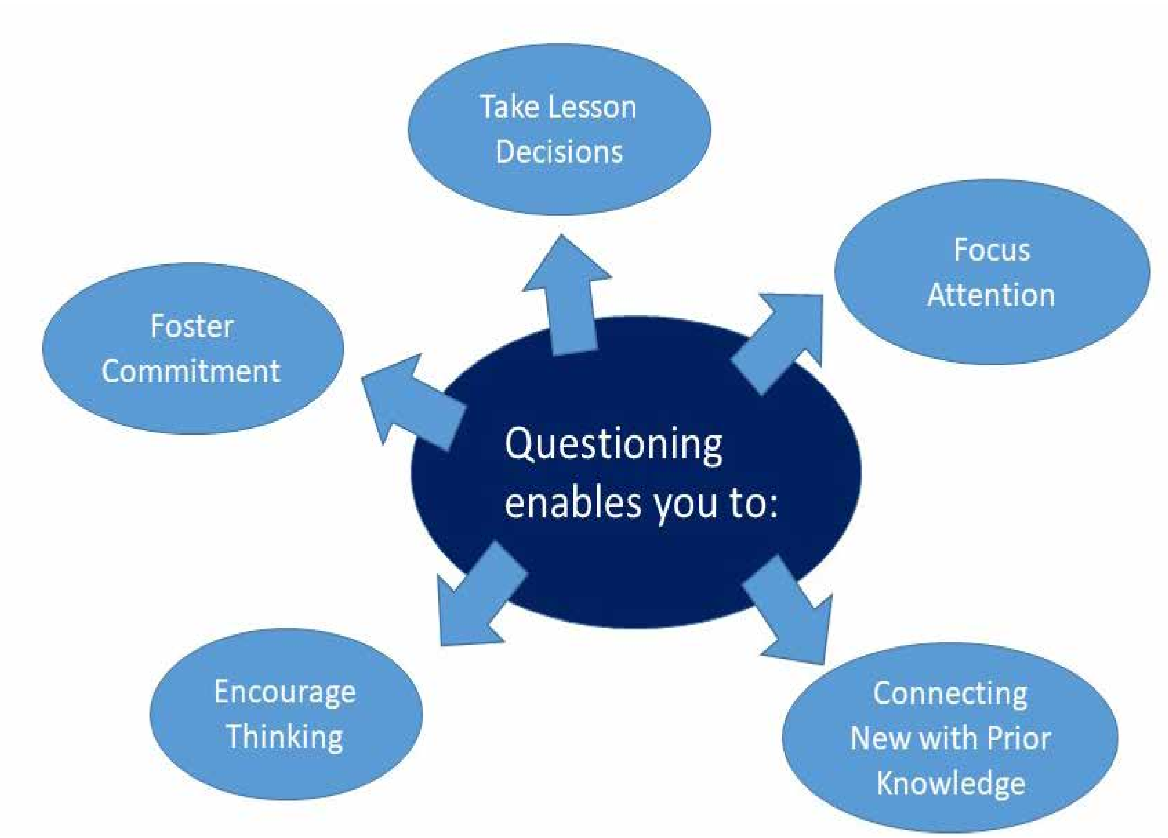


Figure 9: The importance of questioning (VVOB, 2017)

Unfortunately, many teachers don't use the power of questioning to stimulate thinking and learning. Upon hearing a correct answer, they immediately move on. Upon hearing a wrong answer, they correct it or ask another learner to give the correct answer. Some teachers consider a wrong answer as something that needs to be avoided as much as possible.

However, answers of confident students are a bad guide to what the rest of the class is thinking (Wiliam, 2016). In this section, we will underline the **importance of slowing down** and **asking further questions no matter if the response is correct or not**. Questions are not only about getting the right answer from learners but are about developing reasoning skills and the capacity to formulate one's thinking accurately.

The answers of confident students are a bad guide to what the rest of the class is thinking - Dylan Wiliam

Example: What would you say in this situation?

3, 12, 21, 30, …

Teacher: What do you think is the tenth number in this pattern?

Student: I think it's 12

The question above looks like a straightforward growing pattern where 9 is added every time. Many teachers would react on the student's answer by correcting the answer. A better reaction is to ask: "why did you come up with that answer?". This might reveal valuable reasoning. In this example, the learner may have assumed a repeating pattern with 4 units. The purpose of such questions is to create a classroom culture where it is safe to share alternative answers or a different reasoning.

Next, we discuss techniques to help you with effective questioning.

Techniques for Effective Questioning

1. ***Ask Open Questions***

Open questions are questions where the **answers are not limited to a few possible answers**. They are a good way to initiate thinking and start a deeper conversation. Consider "What is 4 + 6?" (closed question) versus "Is there another way to make 10?" (open question) or "How many sides does a quadrilateral figure have?" (closed question) versus "What do you notice about these figures?" (open question).

Open questions help teachers build self-confidence with learners by allowing them to respond at their level of development. Open questions allow for **differentiation**, as responses reveal individual differences. These may be due to different levels of understanding or readiness, the strategies to which learners have been exposed and how each learner approaches problems in general. Open questions signal that a range of responses are expected and, more importantly, valued. By contrast, yes/no questions tend to limit communication and do not provide teachers with as much useful information.

Some **examples of prompts** for open-ended questions are:

- How else could you have …?
- How are these the same/ different?
- What would happen if …?
- What else could you have done?
- Is there any other way you could …?
- Why did you …?
- How do you know?
- Could you use some other materials to …?
- How did you estimate what the answer could be?
- Show me an example of…
- What is wrong with the statement? How can you correct it?
- Is this always, sometimes or never true?
- How can we be sure that…?

Open questions often address higher levels of **Bloom's Taxonomy (**Figure 10**)**. Verbs such as connect, elaborate, evaluate and justify stimulate learners to communicate their thinking and deepen their understanding.

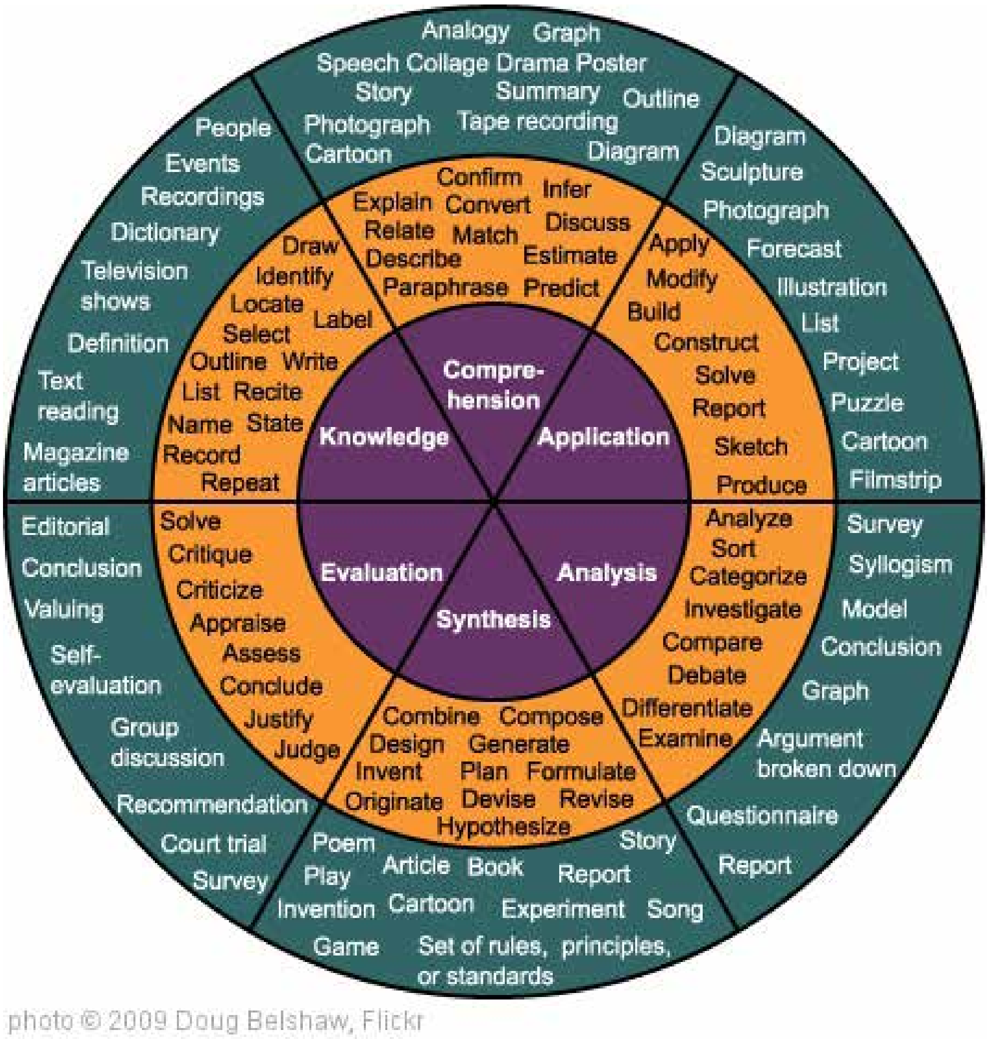


Figure 10: Tools and verbs associated with various levels of Bloom's Taxonomy (Belshaw, 2009)

After asking an open question, it is important to welcome and encourage answers, but not immediately judge them. "Thanks, that is a really interesting answer. Does anyone have something different?" will stimulate discussion, whereas "That is a really good answer." will stop discussion, because learners with other ideas will remain silent. Therefore, judgements should be reserved for the end of a discussion.

Box: Further reading

Resources on Bloom's Taxonomy: http://larryferlazzo.edublogs.org/2009/05/25/the-best-resources-for-helping-teachers-use-blooms-taxonomy-in-the-classroom/

2. Don't let only learners answer who have raised their hands

When you only let learners answer who have raised their hand, it is **easy for other learners not to be involved** (Lemov, 2015). Also, as boys are often more vocal and eager to raise their hands, you risk giving girls fewer opportunities to answer (Consuegra, 2015). It is better to choose yourself who answers a question, or let all learners answer at once raising a card or their hands.

Not focusing on learners who raise their hands (a **"no hands" approach**) has four benefits:

- It allows you to effectively and systematically check for understanding with all learners. You don't just check the students who volunteer. You also want to know how the other students are doing.
- All learners need to think and have an answer ready in case the teacher asks them to respond.
- It increases the pace of questions and answers. You don't ask, "Who can tell me how much is 198 + 65?" and then look around the classroom for hands. You no longer provide hints to get learners' participation.
- It distributes work more equally among learners. It encourages those students who would not volunteer, but know the answer, to participate. It allows you to make sure that boys and girls get equal opportunities to answer.

Example: https://www.youtube.com/watch?v=g-SUzv1t78k

3. Let learners vote with exercise notebooks or voting cards

Exercise notebooks or voting cards on which learners can write answers to questions are a powerful pedagogical tool. After posing the question, the teacher counts down and on 'zero' all learners raise their book or card together. Such exercise notebooks can be useful resources because:

- When learners hold their ideas up to the teacher, he/she can see immediately what every learner thinks.
- During class discussions, they allow the teacher to ask different kinds of questions (typically beginning with 'Show me . . .').
- They allow learners to simultaneously present a range of written and/or drawn responses to the teacher and each other, thereby stimulating all learners to think.

Examples of questions that you can use for the exercise notebooks are:

- Give me two fractions that add to 1. Now show me another pair of fractions.
- Give me a number between 1/3 and 1/4. Now a number between 1/3 and 3/7.
- Draw a quadrilateral with two lines of symmetry.
- You can use multiple-choice questions, where learners write their response (letter) in the exercise notebook.

As a follow-up, teachers can write a few of the learners' answers (anonymously), both correct and incorrect, on the board for discussion with the whole class. When answers are written on the board, learners feel less threatened when the answers are criticised by others. You can let learners vote about what they think the correct answer is and discuss in pairs.

A good way to **use voting** is to let learners evaluate mathematical statements or generalizations. Learners are asked to decide whether the statements are 'always', 'sometimes' or 'never' true, and (important!) give explanations for their decisions. Explanations involve generating examples and counterexamples to support or refute the statements.

Statements can be formulated at any level of difficulty. Some **examples of statements**:

- If you divide a number by 2, the answer will be less than the original number.
- If you divide 10 by a number, your answer will be less than or equal to 10.
- Numbers with more digits are greater in value
- Multiplying makes numbers bigger
- When you multiply by 10, you add a zero
- You can't have a fraction that is bigger than one
- Five is less than six so one fifth must be smaller than one sixth
- Every fraction can be written as a decimal
- Every decimal can be written as a fraction
- If you double the radius of a circle, you double the area.
- Shapes with larger areas have larger perimeters
- A rectangle is also a trapezium
- if you double the lengths of the sides of shapes, you double the area;
- In January, bus fares went up by 20%. In August, they went down by 20%. Michel claims that: "The fares are now back to what they were before the January increase." Do you agree?

In this process, the **teacher's role** is to:

- encourage learners to think deeply, by suggesting that they try further examples ("Is this one still true for decimals or negative numbers?"; "How does that change the perimeter and area?");
- challenge learners to provide more convincing reasons ("I can see a flaw in that argument"; "What happens when …?");
- help learners formulate their thoughts in a mathematically correct way;

4. Let students formulate questions

By asking students to look at some information and think of questions to ask each other, they have to make connections to their prior knowledge. Of course, they will likely start with obvious questions, but with practice they will get more creative. A good practice is to give problems that have gaps in them and ask learners to help you fill in those gaps. The most interesting problems are developed together by teachers and students, not merely assigned by the teacher.

"I like providing students situations with lots of information and asking students to pose the questions we might solve based on this information." (Bushart, 2016)

For example:

- Stimulate learners to formulate questions to introduce the concepts of 'greater, equal or less. The teacher shows two glasses (labelled with 1 and 2) with different amounts of water; then invites learners to pose as many questions as possible of what they are observing.

  Possible questions are:

  - Which glass is greater than another?
  - How much more content does one glass have compared to the other?
  - Which glass has more content? Why?
  - Is the liquid potable?
  - What kind of the liquid is in the glasses?

  Next, the teacher can start a discussion on which questions can be solved mathematically by using numbers and operation.

- The teacher can ask learners to ask questions in relation with learners' presence.

  Possible questions include:

  - How many learners are absent?
  - How many learners are present?

  - How many girls and boys are absent?
  - Are more boys absent than girls?
  - Why are boys/girls more absent than girls/boys?
  - Are more learners absent today than yesterday?

- The teacher presents following information and invites learners to ask questions: Anita has five oranges, Angelique has 20 oranges and Andrew has 15 apples. Elise has no fruit, but has 2000 francs. The price of an apple is 700 francs, the price of an orange is 300 francs and the price of an avocado is 400 francs. What questions can you make from this information?

  Possible questions include:

  - How many apples/oranges/avocados can Elise buy?
  - How many francs can Anita get for her five oranges?
  - How many oranges do Anita and Angelique have all together?

- Anita has five oranges, Angelique has 20 oranges and Andrew has 15 apples. Elise has no fruit, but has 2000 Frw. The price of an apple is 700 Frw, the price of an orange is 300 Frw and the price of an avocado is 400 Frw. What questions can you make from this information?

5. *Not Tennis but Volleyball*

This is a variation on the previous technique. When you ask a question and a learner answers, you can say "correct," and move on. But it's much better to say instead, "Oh, Eduard thinks the answer is 24. John, do you agree or disagree with this answer?" followed by, "Oh, John says he agrees with the answer of 24. Lydia, why do you think both students are saying the answer is 24?" The students' answers pass through you, but you immediately pass them on in the form of a new question to another student in the class. Of course, you don't have to do this if the question is simple.

> *"If I'm teaching P5 graders and for some reason I ask the sum of 12 + 12, then I'm not going to engage in a lengthy discussion, but if the students are evaluating a situation using concepts we're currently working on, then you better believe we're going to talk it out, and they're not going to think the answer is correct because I told them so, but because we built consensus as a class."* (Bushart, 2016)

This will stimulate learners to listen to each other, think actively about each other's responses and develop their reasoning skills.

6. Provide Wait Time after asking a question

Many teachers are uncomfortable with silence. So, they let the first student who raises the hand answer. By waiting a few seconds, several things happen (Lemov, 2015):

- The correctness of students' responses increase.
- The number of failures to respond ("I don't know") decreases.
- The number of students who volunteer to answer increases. Many students simply need more time to formulate their thoughts into words.
- The use of evidence in answers increases.

This can be combined with strategies like **turn and talk** (see: p. 95) and **think-pair-share** (see p. 149**)**, which give learners time to clarify their thinking in pairs or small groups before answering.

Example: https://www.youtube.com/watch?v=dBnuSUL0ymM

7. Right is right: holding out for the correct answer

"Right is right" means that when teachers ask a question, they hold out for a complete answer, or one that would be acceptable on a test, with that student. Students often stop thinking when they hear that their answer is "right." The key idea is that the teacher should set a high standard of correctness by only naming "right" those answers which are completely right (Lemov, 2015). If the answer is not completely correct, the teacher should continue asking questions.

For example,

Teacher: Can you someone give the definition of volume?

Student: The volume is equal to L x W x H

Teacher: That's the formula. I'm asking for the definition.

8. Stretch learners to extend or deepen their answers

Rather than stopping after a student gives you the correct answer, follow up with **questions that extend knowledge and check for full understanding.** You can do this by asking students how they got the answer, what is another way to get the answer, why they gave the answer they gave, how to apply the same skill in a new situation, or what more specific vocabulary they could use. This both challenges students to extend their thinking and checks that students don't get the correct answer by luck, memorisation or partial mastery. This technique sends the message that learning does not end with a right answer. This technique is very important for differentiating instruction (Lemov, 2015).

Prompts or **questions** that you can use **to stretch your students' thinking** are:

- Asking how or why
- Ask for another method to find the answer
- Ask for a better word or a more precise expression
- Ask for evidence
- Saying "tell me more" or "develop that more"

This technique works best when you use it frequently. Avoid using it only when a learner has made a mistake. Learners will quickly realize that you ask these questions to indicate that the learner has made a mistake. You should be asking this question regardless of whether the answer is correct or not.

Example: https://www.youtube.com/watch?v=8P1o8y9ZXWY

Activity 14

Select a topic from appendix 1 and prepare a role play of a teaching sequence in which you apply techniques of effective questioning.

Section 2: Mathematics Conversations

Introduction

A concept that is often used when talking about teaching learners in a language that is not their mother tongue is Content and Language Integrated Learning (CLIL) (Coyle, 2005). This is the case in Rwanda where from P4 onwards, mathematics lessons are taught in English. Many learners are not proficient in English and may struggle to understand mathematical content because they don't understand enough English. Therefore, it is important that all teachers, including mathematics teachers, pay attention to developing the language skills of their learners. This section introduces to you another key aspect to enhance learners' proficiency in mathematics.

Learners often can find right answers to problems but cannot explain how they came up with those answers. By modelling and stimulating discussions and paying attention to using correct mathematical terms, teachers can help learners to express their ideas.

For example, consider the example below about division of whole numbers. Explaining the hypothesis and the proof requires a lot of vocabulary and grammar.

HYPOTHESIS	If a whole number ends in *0* or *5*, then we can divide it by *5* (it is divisible by *5*)
PROOF	*135* ends in *5*, which implies that we can divide it by *5* (which implies that it is divisible by *5*)

Activity 15

Think individually about following questions:

- What elements make a good mathematics conversation?
- How can you stimulate your learners to engage in mathematics conversations?

After a few minutes of thinking, discuss your ideas with your neighbour.

Conversations are a crucial aspect of mathematics lessons. Group work and other student-centred methods are less effective when the quality of mathematical conversations in groups is low (Klibanoff, Levine, Huttenlocher, Vasilyeva, & Hedges, 2006). Students who engage in meaningful mathematical discussions increase their conceptual understanding and deepen their content knowledge. Students also learn to accept one another's ideas. When all students contribute in mathematical discussions, everyone feels that his or her ideas are welcome.

Learners need to produce and practice both **mathematical language** and school language (Bentley & Philips, 2007). Every subject has its own content-specific language. This is the subject-specific vocabulary, grammatical structures and functional expressions that learners need to:

- learn about a curricular subject
- communicate subject knowledge
- take part in interactive classroom tasks.

Secondly, there is the **school language**. This is the non-subject-specific language which learners may have already learned in their English classes and which they can then use to communicate more fully in the subject. For example, mathematics teachers could identify the following language for learning about linear graphs (P5, Unit 14).

Table 2: Mathematical language and school language for teaching about linear graphs

Content-obligatory language	Content-compatible language
linear graph, non-linear graph straight-line graph, curved graph x-axis, x coordinate y-axis, y coordinate the x and y axes I'll plot the coordinates on the graph.	the same, different line, point numbers letters of the alphabet (explaining) This means...

Source: Bentley & Philips, 2007

What can you do to **strengthen learners' language skills during maths lessons**?

1. Revoicing or Paraphrasing

Revoicing or paraphrasing is very useful when a student's explanation is confusing or hard for others to understand. Revoicing means that the teacher repeats all or some of what the learner said and then asks for clarification, which in turn provokes the learner to clarify and offer further explanation. This also gives the teacher an opportunity to embed mathematics vocabulary in the conversation so the learner can further explain his/her thinking (Chapin, O'Connor, O'Connor, & Anderson, 2009, p.14).

An example of a revoicing response is: "*So you're saying that* [it's an odd number?]". When revoicing, the listener repeats part or all the learner's words and asks the learner to say whether the repeated words are correct (Chapin et al., 2009).

Example: https://www.youtube.com/watch?v=X2Oyhrt0hoU

2. Repeating and reasoning

You can stimulate mathematical conversations by letting students repeat or reason, based on another student's answer. Possible prompts are:

- Can you repeat what he just said in your own words?
- Would someone like to add something more to that?"
- What other mathematics content can you connect with this?
- When do you use this mathematics at home? At school? In other places?
- How is this like something you have learned before?

3. Asking why

Tell your learners that "**because" is the magical word** you want to hear in every answer! When they give an answer, they develop the habit of adding "because" and explaining their answer.

Example: Where does 1.6 go on a number line?

- 'I draw a number line that goes from 0 to 2, and I say 1.6 goes here.' (-)
- 'I draw a number line that goes from 0 to 2 BECAUSE ......and I say 1.6 goes here BECAUSE ....' (+)

4. Stimulating precise use of mathematical language

Lack of precise language can confuse students' understanding of a concept and may lead to misconceptions. Using precise mathematical language expands students' mathematics vocabulary. It will also support them in thinking more carefully about their ideas and their peers' ideas.

Examples:

1. Multiplying by 10 = adding a 0 to the right.

Why is putting a zero to the right of the unit not good instruction for multiplication? What can be consequences for students' understanding?

- Adding a zero to the right doesn't work for multiplications with decimal numbers. The same goes for "multiplication makes bigger". This is also not always true. Better is to say: "multiplication makes bigger when/ if ..." "Sharing means less" is also not always true.
- These statements are introduced in early grades and cause confusion in later grades.

2. Meaning of the "=" sign

Learners need to understand that the equal sign shows that quantities on either side of the sign have the same value. However, students often think it means that they should do something to the numbers before it and write the answer after it. They often read an equation like 6 + 1 = 7 as "six plus one *makes* seven."

3. "Moving" terms

Sometimes, teachers use language like moving terms when simplifying algebraic terms. For example, 2x + 5 = x – 4. When solving this equation, they write 2x – x = -4 -5, explaining when moving the term, the sign changes. This can be confusing to learners when they don't understand why the sign changes. Better is to explain the process by adding or subtracting an identical quantity to each side of the equation.

Figure 11 illustrates the importance of being precise with mathematical language

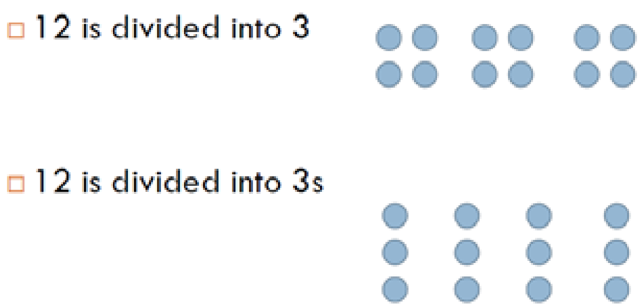


Figure 11: Importance of correct mathematical language in division operations

Why is putting a zero to the right of the unit not good instruction for doing multiplications? The rule doesn't work with decimal numbers. The same goes for "multiplication makes bigger". This is also not always true. It is better to use statements like: "multiplication makes bigger when/ if …"

Activity 16

Think individually about the questions below:

1. Can you find other examples of how unprecise use of language may cause confusion in mathematics?
2. How do you stimulate precise use of mathematical language with your students?

You can promote the use of precise mathematical language with your learners by:

- Use mathematical vocabulary yourself correctly and regularly.
- When planning your lesson**, identify key vocabulary**, and make sure these words are written on the blackboard.
- Introduce and model new vocabulary through explanations, examples, and illustrations.
- Point to symbols when saying the words that the symbol represents.
- When a learner uses a new mathematical word correctly, point it out (for the benefit of the whole class, not just that student).
- Point out when the common definition of a word is different from its mathematical meaning.
- Write new vocabulary on a **Mathematics Words flipchart** and have learners keep their own lists of mathematics words in their exercise books.

Box: Further reading

http://www.edu.gov.on.ca/eng/literacynumeracy/inspire/research/CBS_Maximize_Math_Learning.pdf

http://www.edu.gov.on.ca/eng/literacynumeracy/inspire/research/CBS_AskingEffectiveQuestions.pdf

http://www.nctm.org/Publications/Teaching-Children-Mathematics/2015/Vol22/Issue4/Creating-Math-Talk-Communities/

4. Scaffolding

Just as in construction, scaffolding in the educational context is essentially about support. You need to think about how to bring the content to each student's skill level.

Providing scaffolding, i.e. **content and language support strategies which are appropriate but temporary**, is therefore very important. For example, teachers can write **sentence starters** on the board to support reasoning skills (Figure 12). For example, A ____ angle is an angle that has more than/ less than/ equal to _____ degrees.

We found that the	graph equation	is _____ because _____.

E.G. We found the graph is linear because the coordinates make a straight line.
We found the equation $y = x^2$ is non-linear because the coordinates make a curved graph.

Figure 12: Example of a Scaffolding Structure

Providing effective scaffolding is a challenge because learners vary in the amount of support they need and in the length of time the support is needed. It is useful to think of support **at word and sentence levels.** Table 4 shows some examples for the topic of probability.

Table 3: Examples of word-level and sentence-level scaffolding

Word-level support	Sentence-level support
Word bank: probability impossible not very likely possible likely very likely certain equal chance/equally likely	**Substitution table:** The | probability / chance / likelihood | of _______ is __________ . **Sentence starters:** It is likely that __________ . The probability of it happening is __________ .

Activity 17

Select a lesson topic from the mathematics syllabus and develop a role play for a mathematical conversation on that topic, applying what you learned about mathematical conversations and questioning (section 1).

Section 3: Developing Problem Solving Skills

Introduction

Being able to solve problems is a key objective of learning mathematics (REB, 2015). Solving problems is at the core of what doing mathematics means (Burns, 2015). Learning mathematical rules and facts is important, but they are only the tools with which we learn to do mathematics, not the final objective of teaching mathematics. This section enhances your capacity to in developing problem-solving skills with your learners as a key aspect of mathematics instruction and a generic competence in the CBC.

Problem solving is about engaging with real problems; guessing, discovering, and making sense of mathematics (Polya, 1945). For Polya, problem solving is:

- seeking solutions not just memorising procedures.
- exploring patterns not just memorising formulas.

A three-phase structure for problem-solving lessons (Burns, 2015, p. 135).

A structure in three phases is useful for planning lessons that include problem solving. The three phases are introducing, exploring and summarizing. *Introducing* for launching the investigation, *exploring* for learners to work independently or in groups and *summarizing* for a classroom discussion to share results and talk about the mathematics involved.

Phase 1: Introduction

The goal of the introduction is to help learners understand what they are going to investigate and how they will work. This is best done with the whole class so that everyone gets the same information. You can follow these four steps when introducing an investigation:

- present or review concepts
- pose a part of the problem or a similar but smaller problem
- present the investigation
- discuss the task to make sure that learners understand what they need to do.

Phase 2: Exploring

Once learners understand what to do, they engage with the investigation, usually in pairs or small groups. During the exploring phase, the role of the teacher is:

- observe interactions within groups and help learners to get under way with the investigation;
- assist groups as needed, either when all members raise their hands or when a group is not working productively;
- provide an extension to groups that finish more quickly than others.

Phase 3: Summarising

The summarising part of a problem-solving sequence is very important and should not be skipped or shortened. It is crucial for students to reflect on their learning, hear from others and connect others' experiences to their ideas. To prepare for the summarising, you can let learners write down summary statements about their experiences: what they noticed, conclusions they made etc. Use the summarising discussion to talk about how the solutions can be generalized. Generalising involves extending a solution to other situations.

Example: introducing division grouping problems (Burns, 2015, p. 392)

Victoria and Sam are about to have a snack. Their mother has baked a dozen cookies. Just as they are about to divide the cookies, two friends arrive. The, just before the four children begin to eat cookies, two more friends arrive. And once these six children are about to have their snack, six more friends arrive. Now, there are twelve children and twelve cookies. The children hear that another person arrives, but this time, it is grandmother with a new plate of freshly baked cookies.

Distribute twelve colour tiles or cubes to each group of four students and let them use the tiles or cubes to represent each stage of the story.

After the work in groups, summarize the problem, focusing on:

- mathematical vocabulary: dividend, divisor, quotient
- relation between multiplication and division. Point out the related multiplication equation for each problem. For example, if only 2 children share all twelve cookies, they would each have 6 cookies, which can be represented with multiplication by 2 x 6 = 12. Make sure to read this as "two groups of twelve" and as "two times twelve".

Example: the consecutive sums problem (Benson et al., 2004; Burns, 2015)

The consecutive sums problem is a good example of a problem-solving activity because it challenges learners to investigate patterns between numbers, make hypotheses, test theories and communicate ideas.

During the introduction phase:

1. Present or review concepts: review consecutive numbers.
2. Pose part of problem or similar but smaller problem: use a question such as, *who can think of a way to write the number 9 as the sum of consecutive numbers*? Record on the board, and underneath it, write another equation:

   9 = 4 + 5

   9 = 2 + 3 + 4

This shows that it is possible to write 9 as the sum of consecutive numbers in at least two different ways. You may introduce another example such as 15, which can be represented as the sum of consecutive numbers in three ways:

15 = 7 + 8

15 = 4 + 5 + 6

15 = 1 + 2 + 3 + 4 +5

3. Present the investigation: ask learners in groups to find all the ways to write each number from 1 to 25 as the sum of consecutive numbers (addends). Tell

them that some of the sums are impossible and challenge them to see if they can find a pattern of those numbers. Challenge them to find other patterns as well, such as how many different sums there are for different numbers. Ask groups to write down their findings, equations and patterns.

4. Discuss the task to make sure that learners understand what they need to do.

During the exploration phase:

1. Observe the interactions in the groups, how members divide tasks and what strategies they are using.
2. Assist where needed or when they are headed the wrong way. For example, groups sometimes make erroneous generalisations. They discover that it is impossible to write 2 and 4 as the sum of consecutive numbers and they conclude that 6 would fit the pattern and also be impossible. In such a situation, confront them with a contradiction. Ask the group to consider 1 + 2 + 3. When they realise that the sum of those numbers is 6, you can leave them to rethink their work.

Questions you can ask to stimulate thinking in groups are:

- How could you describe the pattern of numbers that are impossible to write as the sum of consecutive addends?
- What do you notice about the numbers that had three possible ways?
- Which numbers had only one possible way?
- Which numbers cannot be written as the sum of consecutive numbers?

During the summarising phase:

1. Make sure that groups are prepared to report in the classroom discussion.
2. Initiate a classroom discussion about the findings:
   - Start with asking how groups organized the work.
   - Ask what strategies they used. Some groups may have used the guess and

check strategy to find ways that worked. Other groups may have started from writing consecutive addends such as 2 + 3 + 4 and then write each expression under the appropriate sum. It is important to discuss different strategies so that learners become aware that there is often a variety of ways to approach a problem.

3. Have groups report their results, explaining their reasoning or strategies. Discuss any differences and similarities in the solutions.

4. Generalise from the solutions. Ask one group to share one of their statements. Other groups share whether they have found the same statement. Then, a second group shares a statement and so on.

Questions you can ask to stimulate learners to generalise:

- which sums were impossible to write with consecutive addends?
- what patterns did you notice for sums that you could write in two ways? And three ways? Four ways?
- what patterns did you notice for sums that could be written as the sum of two addends? And three addends?
- what do you notice about sums that are prime numbers?
- how you can you link the problem with what you know about multiplication?

More info about the consecutive sums problem:

https://nrich.maths.org/summingconsecutive/solution

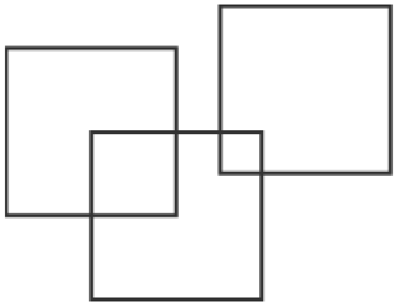
http://mathpractices.edc.org/pdf/Consecutive_Sums.pdf

Example: the three squares problem (https://nrich.maths.org/143/note)

This problem helps learners to develop their understanding of the properties of a square. The interactivity enables learners to access the task immediately, so they can easily begin to explore. This in turn means that they are much more likely to become curious about the challenge of finding as many squares as possible, so are motivated to work mathematically.

The interaction not only supports the exploratory nature of the problem, but also helps to deepen children's understanding of what makes a square a square.

What is the greatest number of squares you can make by overlapping three squares of the same size?



During the introduction phase:

You could begin by arranging just two squares in different ways and asking children to count the number of squares made in each case. You may ask learners to review a square's properties.

Once they are familiar with the idea, introduce the main problem and suggest they work in pairs. You can provide square frames cut from paper/card or made using construction equipment/straws. It would also be useful to have squared paper available for recording.

During the exploration phase:

As in the previous example, you observe interactions in the groups and assist where needed.

As an extension, some learners could try using four squares in the same way, or they could use equilateral triangles instead.

During the summarising phase:

Talk with learners how they went about solving the problem. Did they record as they went along? If so, what and why? You may find that some learners drew an arrangement so that they could count the squares more easily by marking in colour. Others might have recorded an arrangement as a reminder of the largest number of squares they had found so far.

- How many squares can you make by overlapping two large squares?
- How do you know that is a square?
- Can you move the large squares so that you create more squares?
- How do you know that it isn't possible to make more squares?

Example: Measuring round things (Burns, 2015, p. 152)

The goal of this activity is to make learners familiar with the relation between the circumference and area of circular objects.

For this investigation, learners need a range of circular objects (plates, glasses, jar lids, circles cut out of paper…), a ruler or measuring tape and a string for measuring the circumference.

Choose a circular object and demonstrate measuring its diameter and circumference in several ways: measuring with a string, using a measuring tape and rolling he objects over a piece of paper to mark the length of one rotation.

Draw a table on the board with columns: object, diameter, circumference.

Organize learners in small groups and give following instructions:

- draw a table as shown by the teacher.
- choose a circular object and measure its circumference and diameter.
- record the results in the table.
- measure at least 5 objects.
- review measurements to look for patterns that describe the relationship between the diameter and circumference of each object.

After the group work, bring learners together for a class discussion. At some point, focus learners on looking at the result of dividing the circumference by the diameter for each circle they measured.

Next, connect the investigation to pi. Let learners calculate the circumference of any circle by its diameter and add the value in the table. Explain that when you divide the circumference of any circle by its diameter, the result is always a little more than 3. Another way to say this is that pi and the ratio of the circumference to the diameter of the circle. This holds true for all circles, no matter how small or large.

Box: More **ideas for problem solving activities**:

https://nrich.maths.org/primary

http://mathpractices.edc.org/browse-by-mps.html

Activity 18

Prepare a role play in which you teach one of the four examples of problem solving that are discussed in this section. Apply what you learned about questioning and mathematical conversations.

Word Problems

The main method to introduce problem solving in primary mathematics is through word problems. They are often perceived as difficult by learners because of the combination of language and mathematics skills they require. The best way to develop problem solving skills is to expose them regularly to **various types of word problems**. In this section, we discuss strategies for getting the most out of word problems.

Word problems help students to **connect situations to arithmetic operations**. As such, they help students understand the meanings of addition, subtraction, multiplication and division. However, in real-life problems, students will rarely have all the information they need in a nice package in the way most word problems are structured. Instead, students need to collect the data themselves and there is often more than one possible method.

An effective way to develop mathematical understanding is to **present word problems as authentically as possible**. Authenticity does not just mean that the context for the word problem comes from children's daily lives (e.g. dividing candy, buying milk). Authentic word problems have a different structure. All key information is not necessarily included into the problem. In other cases, too much information is available, and students need to select what is relevant to solve the problem. Because of this, students engage with the problem by asking questions, testing ideas, and organizing what they think they need to know.

Almost any question can be used for problem solving (Lemov, 2015). However, a routine problem for one class, may be new for another class. Something challenging for one student can be familiar for another. Therefore, it is important to start from the prior knowledge of students and differentiate. The main criterion of a good word problem is that it should be **non-routine** to the learner.

There are two wrong ways of using word problems. In the first case, a teacher presents learners with a word problem that has all the necessary information already included in it. The learner must read the problem, extract the key numbers and solve (and repeat the same process with the next word problem). Sometimes, this is called the “cookbook” way of solving word problems.

In the second case, teachers provide minimal guidance and let learners struggle with the problem. Some teachers think that fostering struggle with learners helps them with learning. In fact, there is no evidence to support this (Hattie, 2009; Sweller & Cooper, 1985). On the contrary, there is research that shows that this is demotivating for learners and increases inequality by favouring stronger learners.

Strategies to develop problem solving skills of learners

Let’s now look at some strategies to promote learners’ problem-solving skills.

1. ***Stimulate learners to use various problem-solving strategies***

When learners use interesting strategies to solve a problem, teachers should discuss them with the whole class. Explicitly describing and labelling a strategy is a useful way for learners to talk about their methods, learn methods from each other and for the teacher, to provide suggestions. This strengthens student’ belief that their contributions are valuable and that there may be several strategies to solve a problem.

Box: Example problem solving strategies

Show all the ways that fifteen objects can be put into four piles so that each pile has a different number of objects in it.

- what are possible or reasonable strategies to solve this problem?
- which strategy or combination of strategies will you use first?
- did you change strategies or use others as well? Describe.

Box: Example problem solving strategies

Marie and David are playing a game. At the end of each round, the loser gives the winner a coin. After a while, David has won three games and Lisa has three more coins than she did when she began playing the game. How many rounds did they play?

- what are possible or reasonable strategies to solve this problem?
- which strategy or combination of strategies will you use first?
- Did you change strategies or use others as well? Describe.

These are **strategies** that learners may use during problem solving (Burns, 2015):

- Drawing a picture, using a model.
- Looking for a pattern.
- Making a table or chart.
- Using examples to find a general rule.
- Trying a simpler form of the problem (e.g., with smaller numbers). By solving the easier problem, learners may gain insights that can then be used to solve the original problem.
- Try out a possible solution and check if it is correct. Next, narrow down possible solutions and check again.

2. *Let learners ask questions and develop the problem*

You can start a lesson by posting a sentence on the board and ask learners to record the missing information to solve the problem. Only after all students have participated and understand the scenario do you show the question.

Leaving out the question increases participation from struggling students because there is no right answer and no wrong observations. And having a question to solve that students generated increases all students' understanding of the task and their engagement.

Source: http://mathforum.org/workshops/universal/documents/notice_wonder_intro.pdf

For example, consider the word problem below.

> *"Francis has 5 boxes of chocolate bars for his class. Each box has 6 chocolate bars. How many chocolate bars are there altogether?*
>
> You can transform the problem into the following:
>
> *"Francis has boxes of chocolate bars to share with his class."*

How can you use this statement to have a mathematics conversation with your students?

Some possible reactions from learners:

- "We don't know how many boxes of chocolate bars there are."
- "There isn't enough information to know what's going on."
- "We don't know if it is adding, subtracting, multiplying, or dividing."
- "There are multiple people in the class, because it says boxes and share."
- "How many chocolate bars are in each box?"
- "How many boxes did he bring to class?"
- "How many kids are in his class?"

More information: http://www.teachingchannel.org/blog/2016/04/07/math-word-problems/

3. Moving word problems to the start of the lesson

A strategy which is easy to apply is to shift a problem to the beginning of the lesson. Choose a challenging question that summarizes all content that you want to teach and start the lesson with that question. This question serves as a kind of **key question** for the lesson. Let students try out and find the missing information or at least have a clear(er) idea of what knowledge they are still missing. In this way, students will realize why certain knowledge is useful and it will help them to connect new knowledge to prior knowledge. As a teacher, you will get useful information with this technique about what students find difficult and you could give these aspects more information during the lesson.

For example, in a lesson on algebraic reasoning, you could start with this problem:

> *Imagine that you are at a huge party. Everyone starts to shake hands with other people who are there. If 2 people shake hands, there is 1 handshake. If 3 people are in a group and they each shake hands with the other people in the group, there are 3 handshakes.*

How many handshakes are there if there are 4 people? 10 people? Can you develop a rule to help you figure this out for any number of people?

Activity 19

Using one of the techniques described above, develop a word problem that you can use in your class to develop learners' problem-solving skills. Exchange with someone else's problem and review that word problem.

Box: Further Information

Practical ideas for problem solving activities in primary mathematics:

https://nrich.maths.org/10367

http://www.teachingideas.co.uk/subjects/problem-solving

Section 4: Addressing Learner Errors and Misconceptions

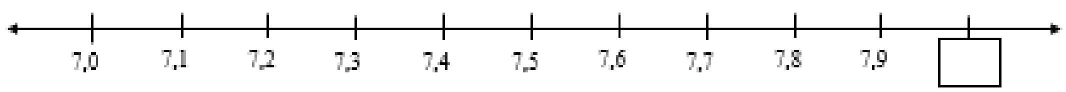
Introduction

This section shows how you can use errors from learners to improve teaching. The reaction of many teachers when students make an error is to correct it as quickly as possible. However, errors can give valuable information about students' thinking. Learners as well need to see errors as opportunities for learning. They must feel that it is okay to offer an idea that might be incorrect and know that they have the support of their teacher and fellow learners to change errors in their thinking.

Errors and misconceptions

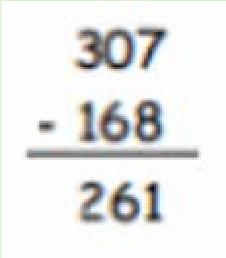
Activity 20

Edouard wrote 7.10 in the empty box on the number line below. Why would he write this? Describe how you could help Edouard to find the correct answer.



Misconceptions are conceptual ideas constructed by learners that make sense to them in relation to their current knowledge (Brodie, 2014). Studies have identified misconceptions for a wide range of mathematical topics (Smith III, Disessa, & Roschelle, 1994). This is a key characteristic of misconceptions: from the student's point of view, they make perfectly sense. Many misconceptions arise from learners' overgeneralisation of a concept from one domain to another (Smith III et al., 1994). For example, from their knowledge of natural numbers, many learners think that adding always means more and that multiplication by 10 is equal to adding a zero to the right. They don't understand that mathematical knowledge that works in one domain (e.g. natural numbers) does not necessarily work in new domains (e.g. decimals and fractions).

This example of student writing comes from the work of Ball and colleagues (2008).



Can you identify the most likely cause of the error in the above learners ’work? How would you address this error?

Activity 21

Consider the following example that was observed in Kayonza. A P6 maths teacher asked learners to solve the following equation: $3(x-2)-(2x+1)=0$. One learner writes on the board: $3x-6-2x+1=0$. The teacher asks the class: “is it correct?”. The whole class responds “yes!” He repeats the question, and the whole class responds “Yes!” Next, he corrects the answer from the learner, pointing out the error.

Discuss following questions:

- Why do you think that the learner made that mistake?
- What do you think about the teacher’s reaction?
- What could you do to address the underlying misconception(s)?

Activity 22

In this exercise, you will discuss some common misconceptions in numbers and operations. For each exercise in the table below, think about an incorrect answer that you have seen your learners make and that reflects a misconception.

1. 3 + __ = 7	2. 35 + 67 -------
3. 42 - 17 -----------	4. 300 - 136 -----------
5. 3840 : 12 (long division)	6. 1/2 + 2/3 =
7. 2.06 + 1.3 + 0.38 =	8. 5.40 X 0.15 ------------------

Following errors and explanations are examples of students' incorrect thinking (Burns, 2015).

Error as a result of a misconception	Underlying misconception
1. 3 + **10** = 7	A plus sign means to add.
2. 35 + 67 ----------- **912**	Add the numbers in each column and write the sums under the line
3. 42 - 17 ------ **35**	When you subtract, you take the smaller number from the larger

Error as a result of a misconception	Underlying misconception
4. 300 - 136 --------- **163 OR 174**	You can't subtract a number from zero, so you change the zeros into nines OR You can 't subtract from zero, so you borrow from the three and the zeros become tens.
5. 3840 : 12 = **32**	You can drop the zero at the end of the problem.
6. 1/2 + 2/3 = **3/5**	When you add fractions, you add across the top and across the bottom.
7. 2.06 + 1.3 + 0.38 = **2.57**	Line the numbers up below each other and add.
8. 5.40 x 0.15 = **81**	After you solve the product, bring down the decimal point.

Thinking about possible causes for these mistakes is important for teachers. Learners who make these errors are not focusing on the meaning of the problem but on the symbols in the problem. For example, they learned that a plus sign means to add, so they combined 3 and 7 to get the sum instead of figuring out how much more was needed to add to 3 in order to get 7. However, the same error does usually not occur when the same problem is presented as a word problem. For example, you have 3 candles, but you need 7 altogether. How many more candles do you need? Learners generally interpret this problem correctly and find that they need three more candles. As a teacher, it is important that learners can explain the meaning of a mathematical problem and not just perform the procedure.

Activity 23

In pairs or small groups, you will receive a curriculum topic from the list below. Come up with two frequent errors that your learners make about that topic that reflect a misconception rather than a calculation error. For each misconception, consider why is the chosen example is a misconception.

Topics:

1. Place value and number sense
2. Addition and subtraction
3. Multiplication and division
4. Fractions
5. Decimals and percentages
6. Geometry
7. Probability and statistics

The following **misconceptions** are **common** in numbers and operations.

1. *Rounding numbers*: When asked to round a value to the nearest 1000, some students mistakenly round to the nearest 10, then the nearest 100 and finally to the nearest 1000.
2. *Multiplication*: Many students think that multiplication always increases the size of a number.
3. *Multiplying decimals*: Mathematical language can be a source of misconceptions. For example, the term "times" is mixed up with "of", thus one-tenth of one-tenth is equal to one-hundredth.
4. *Decimals and their equivalent fractions*: There a misconception that decimals and fractions are different types of numbers while most fractions can be expressed with denominators of 10, 100 or 1000 to find their decimal equivalent.
5. *Dividing whole numbers by fractions*: Many students think that dividing a whole

number decreases its size which is not always the case depending on the type of fraction. For example, dividing three by one quarter, as the number of quarters that fit into three.

6. *Adding with negative numbers*: The term taking away is used to represent minus and giving back for plus. Thus -8 +6 is read as taking away 8 and giving back 6, which is equivalent to taking away only 2. *Taking away* is only one interpretation of subtraction. Subtraction as the difference between two numbers is another one.

Source: https://www.stem.org.uk/elibrary/resource/32755

Box: More information on mathematics misconceptions:

http://www.westada.org/cms/lib8/ID01904074/Centricity/Domain/207/Misconceptions_Error%202.pdf

This document (40 pages) gives an overview of common misconceptions per topic (numbers and operations, fractions, geometry, measurement, probability…).

Techniques to correct learners' misconceptions

Misconceptions cannot be easily "removed" or "replaced" through instruction since they make sense in the light of the learners' current knowledge. As misconceptions arise in the connections between different ideas, the best strategy to deal with them is to understand these connections, rather than to re-teach concepts (Brodie, 2014). Below, we discuss some techniques that can help to address misconceptions without re- teaching the concept:

1. Waiting to give the correct answer, followed by Turn and Talk

A simple technique to address misconceptions is "Withholding the answer". When learners make an error, many teachers immediately correct the learner by asking another learner or by giving the correct answer themselves. By doing this, they miss the opportunity to use such errors as "**teachable moments**". Teachable moments are moments in a lesson that provide great opportunities for learning. Errors can expose valuable information for the teacher, such as misconceptions or incomplete understanding.

Write all answers that learners give on the blackboard. Then, let learners discuss the question in pairs for a few minutes. This is called "**Turn and Talk**". Next, you may let learners vote on what they think is the correct answer. Let learners with different answers explain their reasoning and through questioning, guide them towards the correct answer. Such a discussion will give you information about how many learners have the wrong understanding.

Source: https://onderzoekonderwijs.net/2016/12/10/teach-like-a-champion-8-culture-of-error/

Analyse the root causes of students' mistakes

An important skill as a mathematics teacher is to recognize which learner errors reflect deeper misconceptions. For such errors, it is not enough that learners know that they have made a mistake. They also need to receive feedback on where the mistake lies. Discussing the root causes of mistakes is the best way to change learners' thinking and prevent them from making the mistake again.

2. Concept cartoons

Concept cartoons are simple drawings which put forward a range of viewpoints about the science involved in **everyday situations**.

Figure 13: Concept cartoon about multiplication (Millgate House Education, 2008, adapted by VVOB)

In this example (Figure 13), a situation about multiplication is described. Four cartoon characters talk about what happens when you multiply two numbers. These possible answers are based on research about popular student misconceptions (e.g., Driver, Squires). In this way, the opinions sound familiar to learners and they easily identify themselves with one of the viewpoints.

The main objectives of concept cartoons are **stimulating learners to think, discuss, raise questions, investigate and formulate mathematical arguments** (Keogh & Naylor, 1999). Concept cartoons are more than just multiple-choice questions. They are instruments to trigger student involvement, thinking and exploration. All answers represent frequent misconceptions that occur among learners and should therefore be given attention (why are they incorrect?) during the lesson.

It is very important to notice that all possible answers in the cartoon have equal status. Learners select and specify the best concepts about the situation. This process of conflict between different viewpoints is an important aspect of gaining mathematical knowledge. For this reason, this method is appropriate for learner-centred teaching.

Concept cartoons can reach different objectives (Keogh & Naylor, 1999):

- Making students' ideas explicit (so that potential misconceptions can be identified);
- Challenging and developing students' ideas;
- Stimulating mathematical conversations;
- Providing starting points for exploration;
- Increasing involvement and motivation;
- Deepening understanding of mathematical concepts.

How to use concept cartoons?

Concept cartoons can be integrated through different teaching methods. A possible sequence is the following:

1. Introduce the topic and show the cartoon.
2. Organise a brief period of individual thinking. Collect some quick feedback to see

what range of views is present – perhaps a vote.

3. Encourage discussion in small groups (4-6 learners) and invite groups to reach consensus.
4. Let groups share the outcomes of this enquiry and organize a short whole class discussion, including which alternative(s) seem(s) acceptable and what further information we might need to be sure.
5. Draw ideas together and provide an explicit summary of the initial problem, the enquiry, the outcome and what has been learnt from the enquiry.
6. Consider how students' views might have changed and what has led the change in their ideas.

Activity 24

Review the cartoons in Appendix 3 of your manual. Select one cartoon that is useful for your teaching and prepare a short teaching sequence of 10-15 minutes using the cartoon and appropriate questioning.

Activity 25

Are concept cartoons of any use for your teaching? Place yourself on a continuum between:

- Yes, I will use them often
- No, they aren't of any use

Discuss the pros and contras with the whole group.

Tips for using concept cartoons

Many concept cartoons do **not** have a **single 'correct answer'**. In many cases the only reasonable conclusion involves "***It depends on…***" statements. Even apparently simple situations can have complicating factors when they are examined more closely.

Some teachers find this method of teaching challenging. Their criticism is that the situations in the cartoons should be specified more clearly, so that there is only one right answer. However, it is an important objective of concept cartoons that students learn to express the ideas in the cartoon. It is important to **let students discuss about mathematics**, rather than quickly giving the 'correct' answers.

Therefore, you need to know what possible answers students may give, in order to anticipate the class discussion.

Activity 26

In small groups, develop a short teaching sequence in which you expose and address a misconception in mathematics.

Section 5: Concrete, Pictorial and Abstract (CPA) Approach for Teaching Mathematics

Introduction

Concrete materials are important tools for helping children make sense of mathematics. They can support learning and be effective for engaging students' interest and motivating them to explore ideas (Burns, 2015; Carbonneau, Marley, & Selig, 2013). However, just handing out manipulatives to learners will not make any difference. It is important to understand how concrete materials can help children learn (Van de Walle et al., 2015, p. 30).

Mathematics is a language that uses many representations of ideas. Because of the abstract nature of mathematics, people access mathematical ideas through the representations of those ideas in **symbols** (National Research Council, 2002). There is no inherent meaning in symbols. Symbols always stand for something else. The meaning a symbol has for a child depends on what the child knows and understands about the concepts the symbol represents (Richardson, 2012).

The following symbols may have absolutely no meaning to you. They are inaccessible.

- 你好嗎?
- 我很高興跟你見面

You can compare mathematical symbols with Chinese characters. Without knowledge of the Chinese language, the symbol does not mean anything to you.

Concrete – Semi-Concrete (Pictorial) – Abstract Arrangement

Activity 27

Observe the different representations of seven. Then discuss with your colleagues the questions.

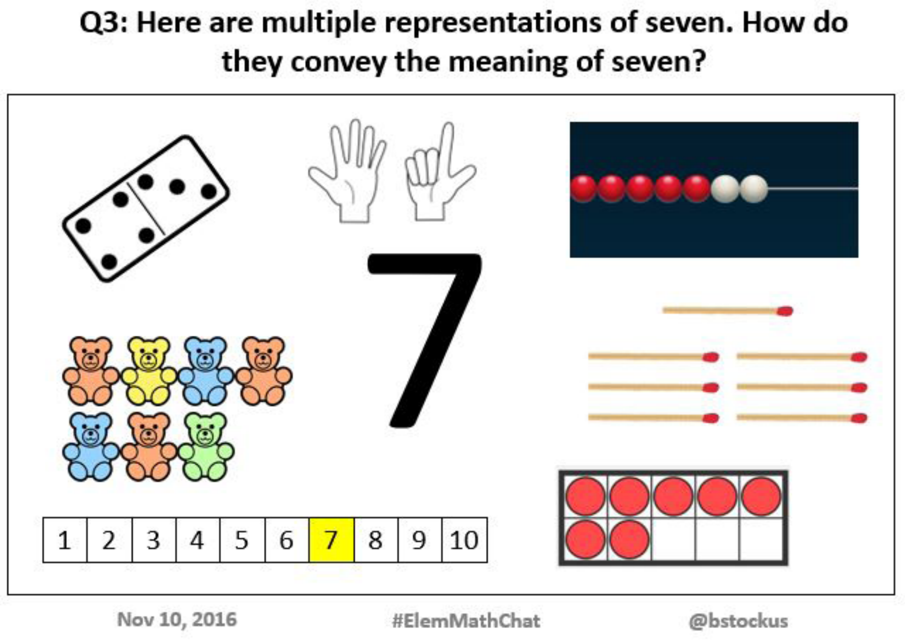


Figure 14: Multiple representations of seven (bstockus)

- What do these representations say to you about the meaning of the number 7?
- Do they all represent the same thing about the number seven?
- Do some representations give you different understandings than others?
- How many different things can you learn about the number seven from these representations?
- How could you make this activity suitable for learners with visual impairments?

For example, the idea of 7 is represented by the symbol "7". *How does the symbol "7" communicate the meaning of seven?*

The symbolic form of this number does not say anything about the number seven. Even if someone told you this is the number seven, what that means to you will vary depending on what you already understand about that number. Just being able to see this symbol and say the word, "Seven," does not necessarily mean someone understands anything about the number seven or the quantity it represents.

Here are some things these representations may say:

- 7 can be made with combinations of smaller numbers: 1 and 6, 2 and 5, 3 and 4.
- You may see a specific combination within a representation, like 4 and 3 in the domino or 5 and 2 in the math rack. After spending time looking at them, do you start to notice multiple combinations within some representations? The teddy bears show 4 and 3 if you look at the rows. However, you might also see 6 and 1 if you look at the group of 6 with 1 teddy bear hanging off the end.
- You may also see that 7 can be made with combinations of more than two numbers: 3, 3, and 1 for example as shown in the matches and the teddy bears.
- The number track shows you where 7 is in relation to other numbers. You can see that 6 is just before 7 and 8 is just after 7.
- You can see how 7 is related to 10. The math rack, number path, and fingers all show that 7 is 3 less than 10.

This is a short list of ways how the meaning of 7 is communicated to demonstrate that the more representations of 7 you give students access to, the better their understanding of the number 7 will become. This applies for any mathematical concept.

Teachers need to provide learners **access to mathematical concepts via multiple and varied representations** and don't rush to the use of a symbol. Without a range of representations, a symbol does not make sense to learners. There is nothing inherently more mathematical about a symbol like 7 than a collection of dots on a domino or seven fingers on my hands. What numeric symbols allow for is efficiency of representing a quantity, especially once the place value system comes into play. But that efficiency is lost on learners, if they do not have a solid foundation in the concepts the symbols represent.

Students with learning disabilities may be weaker in their use of some representations. For these students, it is especially important to use multiple representations. For learners with visual impairments, representations that include touch (using concrete objects) or sound (tapping with the hand) can be used.

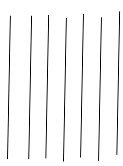
The purpose of teaching through a **concrete-to-semi-concrete-to-abstract sequence** of instruction is to ensure learners develop a deep understanding of mathematical concepts. When students are supported to first develop a concrete level of understanding for a mathematics concept, they can use this foundation to link their conceptual understanding to abstract mathematics learning activities.

Concrete. A mathematical concept is first modeled with concrete materials (e.g. chips, cubes, base ten blocks, beans, pattern blocks). Students are provided with many opportunities to practice and demonstrate mastery using concrete materials.

Semi-concrete (or pictorial). The mathematical concept is next modeled at the semi-concrete level, which involves drawing pictures that represent the concrete objects previously used (e.g. tallies, dots, circles…). Again, students are provided with many opportunities to practice and demonstrate mastery by drawing solutions.

Abstract. The mathematical concept is finally modeled at the abstract level, using only numbers and mathematical symbols. These numbers and symbols are explicitly linked to the semi-concrete representations, so that learners can clearly see what the abstract representations means. Students are provided with many opportunities to practice and demonstrate mastery at the abstract level. If necessary, they can return to the semi-concrete or concrete levels to develop further conceptual understanding.

Example:

Concrete	Pictorial/ Semi-concrete	Formal/abstract
		7

Example: introducing fractions using real-world contexts

Learners need many opportunities to talk about fractional parts, work with concrete materials and relate their experiences to the mathematical notation (Burns, 2015, p. 418). It is important that **fractions as parts of a whole** and **fractions as parts of sets** are introduced.

Use a variety of objects: a bunch of 7 bananas, a set of 12 beans, 5 plastic bottles, a set of 8 bottle caps, some red and some white, and ask questions such as:

- what fractional part is one banana? Two?
- what fractional part are 2 plastic bottles?
- what fraction of the bottle caps is red?
- what fraction of the learners are boys? and girls?
- what fractional part is 4 beans? and 6?

Example: introducing graphs (Burns, 2015, p. 174)

In this Programme, a graph is understood as a mathematical diagram which shows the relationship between two or more sets of numbers or measurements. In the early grades, graphs are best introduced with concrete objects. A pictorial representation of that relationship can be introduced at a later stage and still later, a symbolic representation can be made. The possibilities to make a graph should be taken from the interests of the learners and can draw on experiences in the classroom (Figure 15).

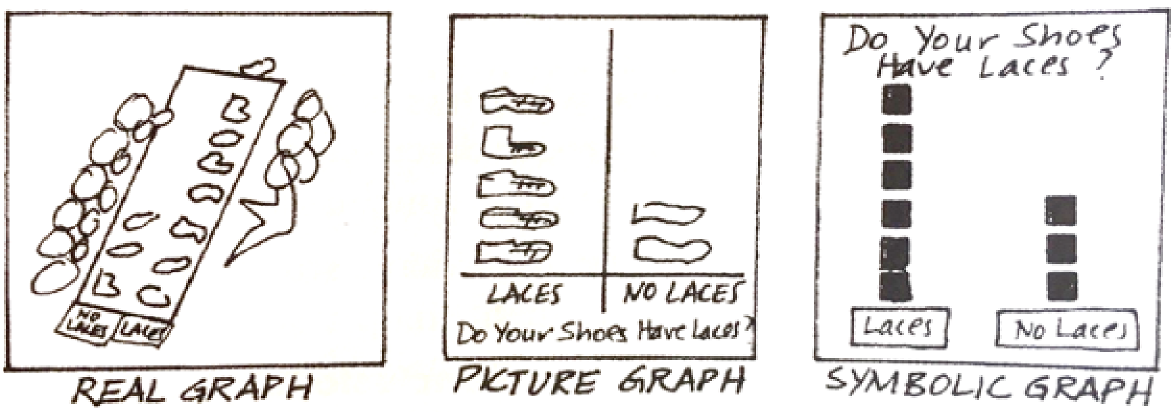


Figure 15: Three main types of graphs (Burns, 2015)

Real graphs use actual objects to compare and build on learners' understanding of more and less. Topics that you can use to make a concrete graph are:

- colours of counters or any other materials
- shoes with and without laces
- male and female learners
- year of birth of learners
- month of birth of learners

Picture graphs use pictures or models to represent real objects. Examples include circles representing counters, drawings of shoes and symbols of people to represent learners.

Symbolic graphs are the most abstract because they use symbols, such as a coloured square or a tally mark, to represent real things.

A variation on introducing graphs is to use a small paper bag and ten tiles or counters in different colours, for example red and blue. Tell learners how many objects there are in the bag, but not how many of each colour. You can put seven or eight of one colour in the bag. Ask learners to take an object from the bag without looking, note its colour, then replace it. Have a learner record the colours on the board using different representations (Figure 15).

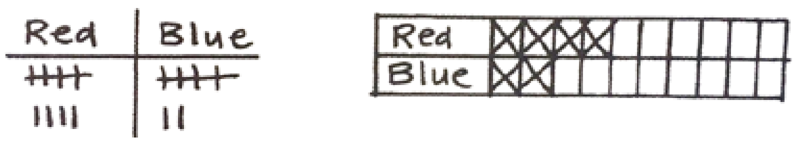


Figure 16: Representation of tiles in the bag activity (Burns, 2015)

After a few drawings, ask learners of which colour they think that there are more objects in the bag. Ask them whether they are sure and when we can be sure about the answer.

An important aspect of a lesson on graphs is **the discussion and interpretation of the information**. You can use following questions to discuss:

- which column has the least/ most?
- are there more/ fewer …?
- how many more/ fewer are there …?

Example: Fraction Kits

A fraction kit introduces learners to fractions as parts of a whole. Learners can develop their own fraction kit. To make a fraction kit, you need five strips of paper (approx. 7 cm x 40 cm). If possible, use thick paper such as Manila paper and use for each strip a different colour or let learners colour each strip. Each learner should have an envelope to keep the strips.

Give each learner a set of five strips and provide directions to cut and label them. They leave one strip whole and cut the others into halves, fourths, eights and sixteenths. Decide on which colour to use for each strip so that all the fraction kits are the same.

Choose a colour and model for the learners how to fold it in half, open and label each sections 1/2, cut on the folds so they have two pieces and write the learner's initials on the back of each piece. This will be helpful when pieces get misplaced. Review the rationale for the notation ½ by explaining that they divided the whole into two sections of the same size, that each piece is one of the two sections and that ½ means one of two equal pieces. Next, choose a colour for the second strip and model for the students how to fold it in half and then half again, open and label each section ¼, cut on the folds so they have four pieces and write their initials on the back of each piece. Talk about each piece being one of four, or one-fourth. Next, repeat the process for 1/8 and 1/16.

Each learner now has a fraction kit to use. Having learners cut and label the pieces helps them relate the fractional notation to the concrete materials and compare the sizes of the fractional parts. They can see that ¼, for example, is larger than 1/16 and they can measure to prove that 2 of the 1/8 pieces are equivalent to ¼.

You can **extend the set with other fractions**, such as 1/3, 1/6, 1/9 and 1/12. You can make a fraction die with the faces labelled 1/2, 1/3, 1/4, 1/6, 1/6 and 1/12 to play Cover and Uncover and record equations (see: Section 6 Games).

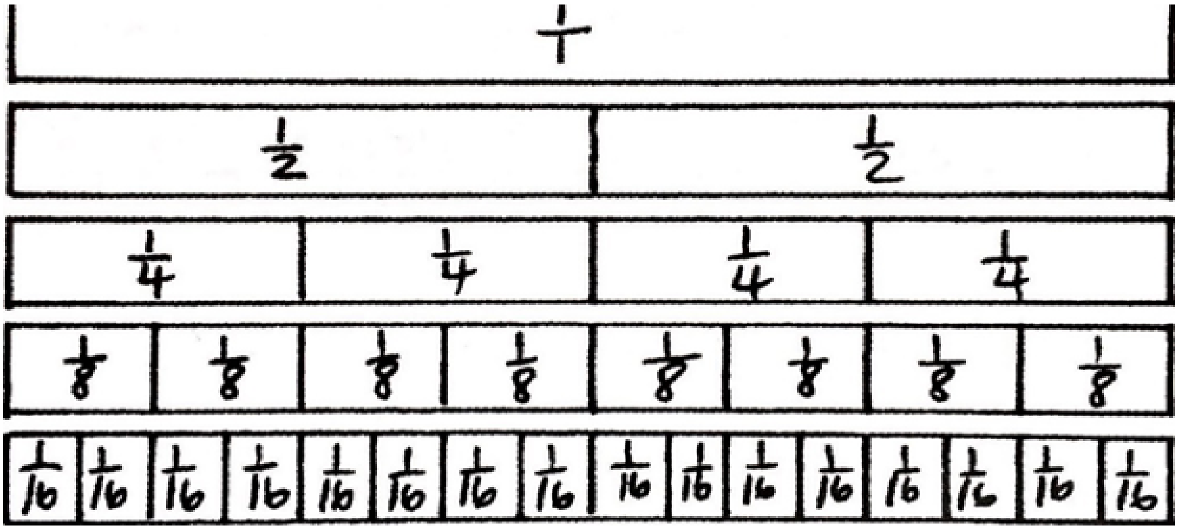


Figure 17: Example of a fraction kit (Burns, 2015)

Example: Growing Patterns

You can find a description of the activity in the Appendix (Section 3: Elements of Algebra).

Activity 28

Think individually about the questions below. After about 10 minutes, discuss your ideas in small groups.

- Identify existing teaching resources for mathematics in your school. Classify these resources (physical/concrete, pictorial).
- Which resources do you find particularly useful and why?
- What resources can you make yourself or find in your environment?

Activity 29

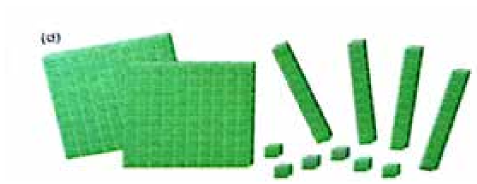
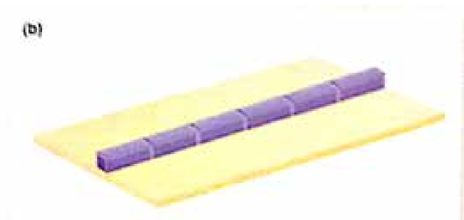
Work in small groups and review the resources listed in Table 5 and Table 6. For which topics are these materials particularly useful? Do you use other materials in your lessons?

Give examples of how you use these or other materials, as in the example below.

Examples of low-cost resources

There are many low-cost resources that are useful for the primary mathematics teacher.

Table 4: Examples of physical models of mathematical concepts (Van de Walle et al., 2007, p. 32)

	
Countable objects can be used to model "number" and related ideas such as "one more than". They are useful to explain place value and decimals.	Base-ten concepts (ones, tens, hundreds) are frequently modelled with strips and squares. Sticks and bundles of sticks can also be used.
	
"Length" involves a comparison of the length attribute of different objects. Rods can be used to measure length.	"Chance" can be modelled by comparing outcomes of spinners with various colours.

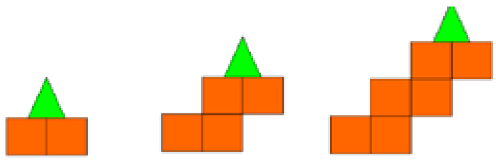
	
Number track/path: no zero, shows counting numbers, ideal for young children because it shows 'distinct steps' that they can count. Precursor to the number line.	Fraction kits are useful to help learners understand the relative sizes of fractions.
	
Cards in different colours and shapes can be used to introduce patterns	Toothpicks (or mud sticks) and rubber bands can be used to teach place value

Table5: Examples of low-cost teaching resources for primary mathematics

	Counting and number equipment made from bottle tops		More place value equipment: hundreds, tens and ones made from bottle tops
	Place value equipment, hundreds, tens and ones made from cardboard boxes		

Source: BLF, 2019

Activity 30

Develop a short teaching sequence in which you move from a concrete to a pictorial stage and introduce the abstract concept. Pay attention to:

- Explicitly linking the different stages;
- Differentiation: some learners may need more time or opportunities in the concrete or pictorial stage.
- Use good questions (open questions, thinking questions) and conduct mathematical conversations.

Section 6: Games

Introduction

Games can be very useful to capture learners' interest and provide alternative ways for engaging them in learning mathematics. Games are also ideal for letting learners work independently and productively. Games can address various skills such as strengthening procedural knowledge, strategic thinking and creativity.

Good mathematics games for the class should be:

- Easy to teach;
- Accessible to all students;
- Reinforce understanding and/or provide practice;
- Encourage strategic thinking;
- Rely only on a few materials;
- Can be played at different levels for differentiation.

Examples of Games

1. Four Strikes and You Are Out (Burns, 2015, p. 89)

This game helps learners to practice numbers and operations. You can play the game at different levels, choosing numbers and operations that are appropriate for the level of your class.

First, explain the game by playing it with the whole class.

Write on the blackboard:

___ ___ + ___ ___ = ___ ___   0 1 2 3 4 5 6 7 8 9

Explain that each blank contains one number and the purpose of the game is to find the numbers in the problem. A learner guesses a number and if it is in the problem, you write it in all the places where it belongs. If the learner guesses a number that is not in the problem, he/she gets a strike.

For example, take the sum, 35 + 10 = 45

The first learner guesses 3, so the teacher writes:

3__ __ + ___ ___ = ___ ___   0 1 2 ~~3~~ 4 5 6 7 8 9

The next learner guesses 2, so the teacher writes (X means 1 strike):

X

3__ __ + ___ ___ = ___ ___   0 1 ~~2~~ ~~3~~ 4 5 6 7 8 9

The next learner guesses 9, so the teacher writes:

XX

3__ __ + ___ ___ = ___ ___   0 1 ~~2~~ ~~3~~ 4 5 6 7 8 ~~9~~

The next learner guesses 5, so the teacher writes:

XX

3__ _5 + ___ ___ = ___ 5_   0 1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 ~~9~~

Now, there are some clues in the problem, that can help learners to make the next guess. You can ask learners to briefly discuss in pairs what number should be guessed next. Some learners may realize that the two 5s means that there had to be a zero in the ones position of the second number. After a minute, ask a learner to guess the next number.

If the learner guesses a zero, ask why. As learners play the game a few times, they start to reason numerically about how clues can help. This requires mental maths skills and develops their number sense.

XX

3__ _5 + ___ _0_ = ___ 5_   ~~0~~ 1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 ~~9~~

Again, you can let learners discuss what numbers could work and which ones are impossible. For example, 8 is no longer possible. The next learner guesses 7.

XXX

3__ _5 + ____ _0_ = ____ 5_ ~~0~~ 1 ~~2~~ ~~3~~ 4 ~~5~~ 6 ~~7~~ 8 ~~9~~

Discuss with learners the remaining possibilities: 1, 4, 6 and 8. Which ones are possible? Learners may find out that the remaining numbers are 1 and 4. In that case, they solved the problem with only 3 strikes, so they won the game.

You can repeat the game with other examples, such as 50 + 26 = 76 and 29 + 13 = 42. Later, you can move to 3 digits and include subtraction, for example 37 + 87 = 124 and 70 – 12 = 58. You can also introduce problems that involve multiplication and subtraction.

When learners understand how to play, they can learn to play the game independently. Let pairs of learners play against other pairs to encourage discussion among learners. Having learners play in pairs allows for both cooperation and competition.

2. Numbers and operations game

This activity can be used as a game to practise learners' skills in basic operations. You can make the sequences as difficult as you like.

Given a set of 5 numbers, try to get as close as possible to the number on the top by using addition, subtraction and multiplication with the numbers below:

30	**45**	**61**
9	9	9
1	6	8
3	11	7
7	3	3
4	2	11

3. Seven Up (Burns, 2015, p. 93)

This game helps learners to develop fluency with combinations of 10. For this game, you need 40 cards, each numbered 1 to 10. To play, learners deal 7 cards faceup in a row. They remove all 10s, either individual cards with the number 10 on them, or pairs of cards that add to 10, and place those cards in a pile separate from the deck. Each time they remove cards, they replace them with cards from the remaining deck. When it is no longer possible to remove any cards, they deal a new row of 7 cards on top of the ones that are there, covering each of them and any blank spaces with a new card. When those cards are removed, it is possible to use the cards underneath. The game ends when it is no longer possible to make 10s or all the cards in the deck are used up.

First, play the game with a few learners in front of the class. When learners understand how to play the game, they can play it in small groups. One learner has the job of removing cards and the other puts out the 7 cards to start and adds new cards to fill the spaces or when they are stuck.

4. The Greatest Wins (Burns, 2015, p. 94)

This is another game to practice learners' skills in basic operations. It can be adapted for various grade levels. For this game, learners need a die with the numbers 1-6 on it.

You start the game by drawing a game board for each player on the blackboard, for example:

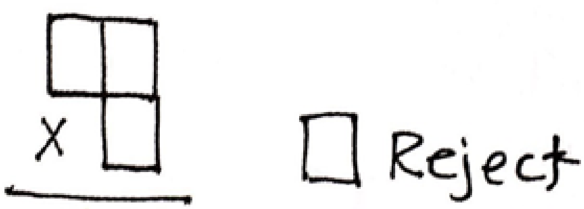


Figure 18: Game Board example for The Greatest Wins game (Burns, 2015)

Learners take turns rolling the die and writing the number in one of the boxes on the game board. Once a learner writes a number in a box, that number can't be changed. Students use the "reject" box to write one number that they think is not helpful. After all players have filled the boxes, learners do the calculation and compare to see who has the greatest answer.

Introduce the game by asking two volunteers to come up to the front of the class and play the game with the teacher. When students understand the game, they can play it in small groups.

After playing a few rounds, organize a discussion about the strategies that learners use to play the game, asking questions such as: How did you decide where to place a 1 or 2? What about a 5 or 6? Who has a different idea about where to place those numbers?

Below are some **variations on the game board** that you can use. Notice that for the first game board, no computation is needed. For this game setup, it is important that students read the resulting number aloud. You can also change the game into *the smallest wins*. Instead of using a die, you can also use a spinner with the numbers 1-9.

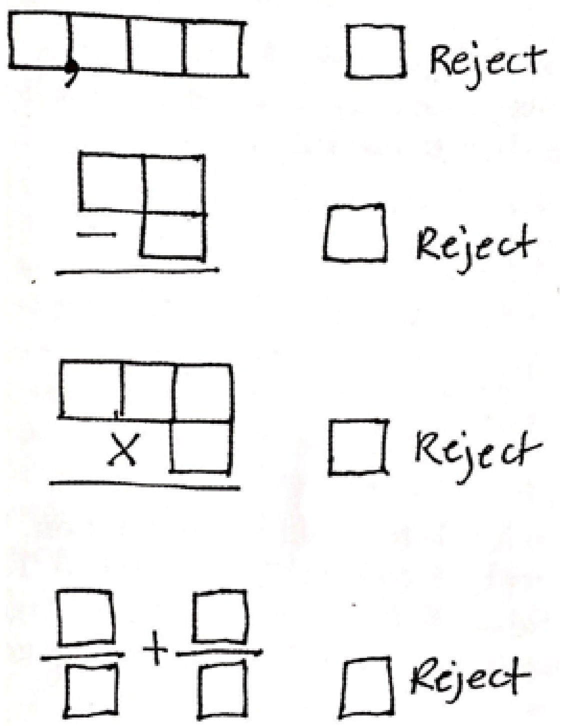


Figure 19: Game board variations for The Greatest Wins game (Burns, 2015)

5. Target 300 (Burns, 2015, p. 96)

This game develops learners' number sense, gives them practice in multiplying by 10 and multiples of 10. The objective of the game is to get a total closest to 300 after six rolls of a 1-6 number die. The total can be exactly 300, lower than 300 or higher than 300 but players must use all six turns.

The first player rolls the die and decides whether to multiply the number that comes up with 10, 20, 30, 40 or 50. Learners record their own and each other's problems. For example, if player 1 rolls a 2 and multiplies it by 20, both players record 2 x 20 = 40. Then player 2 takes a turn. Players keep a total of their scores. After each player has had 6 turns, they record the following:

_______________ won

_______________ was _____ points away from 300.

_______________ was ______ points away from 300.

Again, you can make variations to the game depending on the grade level or as a way to differentiate within your class. For some learners, you can change the game into *target 200* and for others, you can make it *target 600.* Instead of a die, you can use 2 dice or a 1-9 spinner to increase the range of possible numbers.

6. Leftovers (Burns, 2015, p. 410)

This game helps learners to practice division. You need two 1-6 dice per group of students to play the game. The goal of the game is to get the highest possible score. Play the game using the following rules:

1. Agree on a starting number between 200 and 500.
2. One player rolls the two dice and uses the number to make a two-digit divisor. For example, if the learner throws a 3 and a 5, he/ she can use 35 or 53 as the divisor. The learner divides the starting number by the divisor and keeps the remainder as his/her score.
3. The other player records the division sentence, marking the division sequence with the first player's initial.

4. Both players subtract the remainder from the starting number to determine the next starting number.
5. Learners change roles and repeat steps 2-4.
6. Continue switching roles and playing until the starting number becomes zero or it is no longer possible for either player to score.
7. Calculate the total remainders for each player. The player with the greater total is the winner.

7. Hit the Target (Burns, 2015, p. 412)

This game helps learners to practice multiplication. To play the game, learners need one 1-6 die per pair. the goal of the game is to hit the target range in as few steps as possible. Play the game according to following rules:

1. To choose a target range, throw the die three times (or four times to play with greater numbers). Arrange the three or four numbers into the highest possible number. This is the lower end of the target range. For example, if you roll 3, 2 and 6, then the number you make is 632. Add 50 to the original number (or a smaller number to make it more difficult) to determine the upper end of the target range. In the example, the target range becomes 632-682.
2. Player 1 chooses a number between one and hundred (14 for example).
3. Player 2 chooses another number between one and hundred to multiply the first number with, for example 50.
4. If the product doesn't hit the target range, player 2 goes back to the original number (14 in this example) and multiplies it by another number. Player 1 verifies and records the result.
5. Players repeat step 4 until the product falls within the target range.
6. Learners repeat the game, switching roles.

8. Cover up and Uncover (Burns, 2015, p. 424)

This game enables learners to practice simple fractions. For this game, learners need their fraction kit (see Section 5), a coloured whole blue strip per player and a die with the faced labelled: 1/2, 1/4, 1/8, 1/8, 1/16 and 1/16. Learners can play the game following these rules:

1. Both players start with a blue whole strip to cover up.
2. One player rolls the fraction die. The fraction on the die tells what size piece to place on the whole strip.
3. The player gives the die to the other player, who now rolls the die and repeats the process.
4. continue until a player has completely covered the strip with no overlaps. If you roll a fraction that is too big, you give the die to the other player to throw.

A variation on this game is called **Uncover**. For this game, learners need a fraction kit and the same fraction die as in the Cover Up game. Play the game according to the following rules:

- each player starts with the whole strip covered with the two 1/2 pieces. The goal of the game is to uncover the whole strip completely.
- one player rolls the fraction die. The fraction on the die shows what size piece to remove from the whole strip. There are three options: remove a piece (only if there is a piece that is the size indicated by the fraction die), exchange any of the pieces left on the strip for equivalent pieces or do nothing and pass the die to the next player.
- the next player throws the die.
- continue until a player has uncovered the whole strip, without any overlaps.

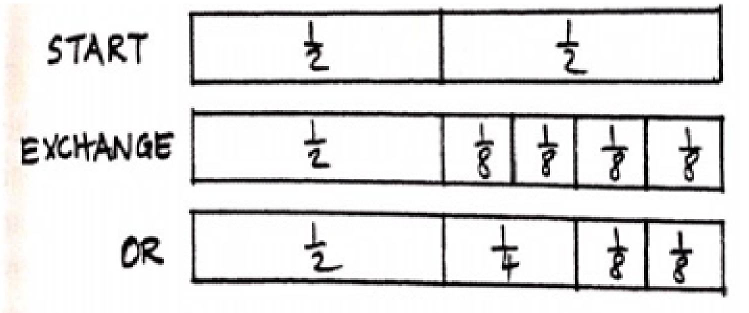


Figure 20: Overview of the Uncover game (Burns, 2015)

Stimulate learners to use correct mathematical language during this game. On the board, record some examples. For example, if a learner used three 1/4 pieces and two 1/8 pieces, write: 1/4 + 1/4 + 1/4 + 1/8 + 1/8 = 1. Next, explain how to shorten the equation by counting the fourths and the eighths and writing: 3/4 + 2/8 = 1

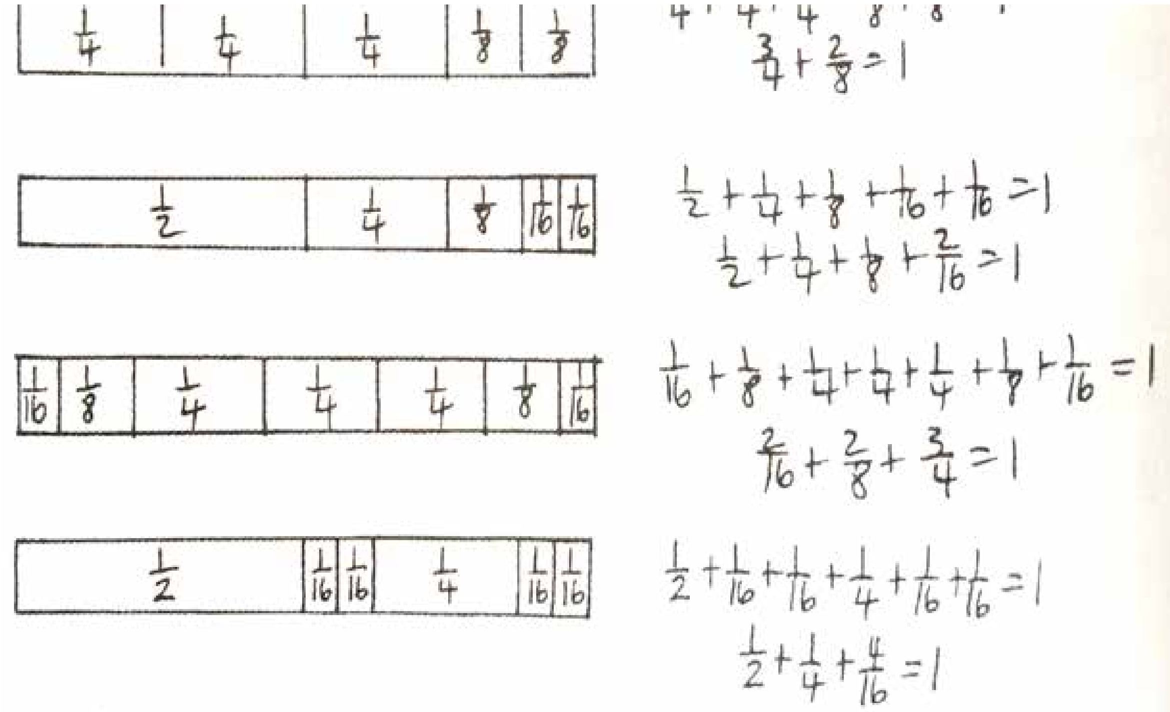


Figure 21: Example of learners' recordings from the Uncover game (Burns, 2015)

9. Circles and Stars (Burns, 2015, p. 378)

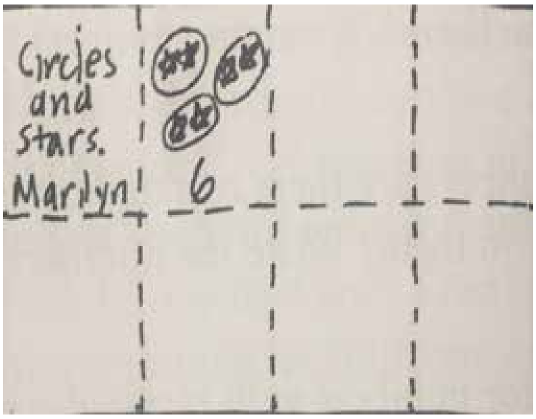
This game introduces learners to multiplication as combining equal groups (repeated addition interpretation of multiplication). Students move from a pictorial representation (drawing circles and stars) to a symbolic representation (writing and reading the equations).

You need one 1-6 die per pair of students, a piece of paper and a pencil. Students play the game in pairs.

Rules of the game:

- Fold a piece of paper in 8 sections. Let students write their name in the first section and use the other sections to draw circles and stars.
- Roll the die. Draw that many circles.
- Roll the die again. Draw that many stars in each circle.
- Record the total number of stars that you drew.
- Give the die to the other player.
- Continue playing until you have drawn circles and stars in each section of the paper.
- The player with the most stars and circles drawn on the whole sheet is the winner.

During the class discussion, draw a sample page of three circles with two stars in each and underneath write: 3 x 2. Explain to students that this is a way to write three groups of two with maths mathematical symbols. Tell them that you can also read it as “three times two” and it means the same thing. Write = 6 and explain this complete the equation to tell how many stars there are in all for that round. Write on the board the different ways to read 3 x 2 = 6.



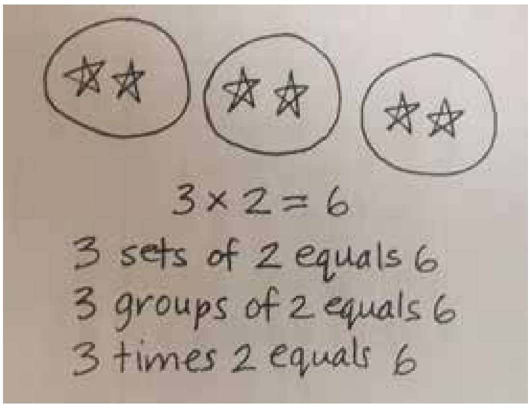


Figure 22: Circles and Stars Game

Let students work in pairs to write a mathematical equation for each section of their paper.

Use following questions to help students understand multiplication:

- Can you find two rounds with the same total but with different arrangements of circles and stars?
- What is the smallest number of stars possible in one round? What is the greatest number?
- Who drew twelve stars in one round? Describe how many circles you drew and how many stars you drew in each?

The purpose of this game is to teach the concept of multiplication, not to let them practice multiplication facts.

Activity 32

In groups of 4, try out one game and play it with your group.

In the second phase, we will mix the groups. Explain the game that you studied to the other group members and play the game.

Below we list some **tips** for using mathematical games:

- It is good to have a **mix of competitive and cooperative games** throughout the year. Competitive games help students test their skills take risks and learn to be graceful winners and losers. However, it is also important to develop communication and cooperation among students. Having students play in pairs or small groups allows for both cooperation and competition.
- Often, you can play the game first with the whole class, so all students are familiar with the rules. Play the game step by step and say aloud what you are doing and why. Next, they can play the game in small groups.
- Encourage learners to play the games at home.
- Games are ideal for engaging learners in your class, freeing up time for you to work with learners who require additional support.

Section 7: Group Work

Introduction

Research has shown that small group work has positive effects on social skills and mathematics learning, but this effect is dependent on (1) shared goals for the group and (2) individual accountability for the achievement of the group (Askew & Wiliam, 1995). This section discusses what group work is, when to use it and how to use it in your lessons.

Activity 25

Why do you use group work in your lessons? Are there situations that group work is not useful? Explain your ideas.

There is a clear difference between working in a group and working as a group (Swan, 2005)

Conditions for Successful Group Work

Learners working in groups is a key component of learner-centred pedagogy. However, group work is not always appropriate. When the purpose of the lesson is to develop fluency in a skill and there is little to discuss, then individual practice is more suitable. Group work is useful when the purpose of the lesson is to develop conceptual understanding or problem-solving skills. In these cases, learners need to share their interpretations and approaches.

There is a clear difference between working **in** a group and working **as** a group (Swan, 2005). It is common to see learners working independently, even when they are sitting together. Sometimes, one group member does the work and others copy the solution. In this case, learners work in a group, but not as a group.

Students need practice, discussion and encouragement to learn to work productively in a group. Critical sharing, participation, listening and communication skills include:

- allowing all members in the group to express their ideas,;
- overcoming shyness and being willing to cooperate with the group;
- listening rather than simply waiting to offer one's own point of view;
- taking time to explain and re-explain until others understand.

Advantages of group work

Research into group work (DfES, 2004; Stewart, 2014) highlight many benefits for both teachers and learners.

For **teachers**, it can help them to:

- empower learners in group situations to engage in peer teaching, learning and assessment to show what they know, understand and can do and identify what they still have to learn;
- get information about how learners are understanding and applying the learning content.

For **learners**, collaborative learning can help them to develop their thinking and problem-solving skills by encouraging them to:

- explain and negotiate their contributions with others in a group;
- take turns in discussion while exploring a topic;
- apply their knowledge to practical situations;
- develop mathematical language skills:
- support and build on each other's ideas.

Additional benefits of group work include:

- Academic achievement: Research has shown that students who work in cooperative groups do better on tests, especially regarding reasoning and critical thinking skills.
- Motivation: One reason for improved academic achievement is that students who are learning cooperatively are more active participants in the learning process. They care more about the class and the material and they are more personally engaged.
- Life Skills: teamwork is essential in modern workplaces. Group work helps them to develop argumentation and listening skills.

Role of the teacher during group work

What should teachers do during group discussions? Here is some guidance (Mercer & Sams, 2006):

1. *Make the purpose of the task clear*

Explain what the task is and how learners should work on it. Also, explain why they should work in this way. For example, "Don't rush, take your time. The answers are not the focus here, but the reasons for those answers. You don't have to finish, but you do have to be able to explain something to the whole group."

2. *Set clear rules for group work*

It helps to prepare students to work together by establishing rules. These rules can be very helpful:

- Learners must be willing to help any group member who asks. When someone asks a question, don't just give the answer, but help by asking questions that helps the learner focus on the problem at hand.
- You may only ask the teacher for help when everyone in your group has the same question. This rule forces learners to discuss questions first among themselves. It motivates learners to rely more on each other and less on the teacher.

3. *Divide the groups*

There is no optimal group size. It depends on the students and the task at hand. The most important thing is to make sure and monitor that all learners are involved. It is good for learners to have the opportunity to work with all their classmates over the course of time and therefore it is best to change group composition regularly.

Mixed-ability groups should be used whenever possible. This promotes co-operation, peer support and valuing individual contributions and is especially useful for project work, learning or practising a new skill, discussing an assignment, problem solving etc. Same ability groups can help you focus on developing a skill or concept with learners, especially when you are differentiating in terms of content or learning processes. Same ability groups should only be used on a temporary basis and should not be composed of the same learners all the time.

In mixed-ability groups, members can help each other with the content of the lesson or with the language of instruction. However, you need to **stimulate that stronger learners support the weaker learners**. You can do this in various ways:

- game as an example to stimulate collaboration within heterogeneous groups
- give each group member let each member of the group solves a question. Building part of the house for each good answer.
- give each group member a number. Call out a number and the learner with the chosen number needs to present the results of the group work.

Grouping by ability may lead to labelling learners and place them always in the same low, middle, or high group. However, in some cases ability or homogeneous grouping can be useful:

- Students are not forced to wait or rush: When you place students of the same ability together, they usually are able to work at about the same pace. This means the students that understand the concept you are teaching can move on to a more advanced stage and the ones that need extra guidance can slow down and get extra help.
- Teachers can work more intensely with those that need help: since they are seated and working together, you can take this opportunity to sit with the ones that need extra instruction.

4. Listen before intervening

When approaching a group, stand back and listen to the discussion before intervening. It is all too easy to interrupt a group and give the right answer.

The purpose of an intervention is to increase the depth of reflective thought. Challenge learners to describe what they are doing (quite easy), to interpret something ("Can you tell us what that means?") or to explain something ("Can you tell us why you said that?").

When a learner asks the teacher a question, don't answer it immediately, but ask another member of the group to answer.

Techniques for effective group Work

1. Think- Pair-Share

In a **Think-Pair-Share** approach means that learners first work alone, writing down their ideas or solutions, then pair and exchange ideas with a partner. Finally, the sharing is done during the class discussion.

Video: https://www.teachingchannel.org/videos/think-pair-share-lesson-idea

In a think-pair-share, learners begin by responding to a task or question individually. Usually, this does take only a few minutes. By letting students first think and prepare the question individually, you ensure that everyone can contribute to group discussions.

In pairs, learners can provide each other with a different explanation or perspective. In some cases, you can join pairs together into fours so that a broader consensus can be reached. Each pair chooses one item to share with the whole group. Quickly go around the room hearing each pair's items. Finally ask, "Did anyone have any other findings they wanted to share?" and collect those. In this fashion, each student is stimulated to think before hearing from others, and students who are thoughtful and move slowly get a chance to organize their thoughts before sharing. Finally, collect some examples of different responses and write these on the board anonymously.

Further reading:

http://mathforum.org/workshops/universal/documents/notice_wonder_intro.pdf

https://www.cultofpedagogy.com/think-pair-share/

2. Talking Points

Talking points are an effective method to stimulate mathematical conversations in groups (Lemov, 2015). You can use the technique at the beginning or end of a lesson to collect prior knowledge or review the topic. You prepare a set of statements that reflect the lesson objectives or misconceptions about the topic.

During the group discussions, learners follow a fixed routine:

1. Go around the group, with each person saying in turn whether they agree, disagree or are unsure about the statement and why. Even if you are unsure, you must state a reason why you are unsure. No comments on each other's answers are given. You can change your mind during your turn in the next round.
2. Go around the group again, with each person whether they agree, disagree or are unsure about their own original statement or about someone else's statement they just head and say why. No comments on each other's answers are given. You are free to change your mind during your turn in the next round.
3. Take a tally of agree, disagree and unsure and make notes on your sheet. No comments are given.
4. Move to the next talking point.

Consider this example of a talking points sheet on fractions:

Talking Point	Agree	Disagree	Unsure
1. Fractions are always less than 1.			
2. A fractions is a number.			
3. We can locate fractions on a number line.			
4. Fractions tell us a size.			
5. One half is always greater than one third.			
6. We can combine fractions.			

Figure 23: Talking Points on fractions

Each statement refers to a specific lesson objective. You can follow up on the group activity with a class discussion. Questions you can use are:

- Which talking point did your whole group agree with and why?
- Which talking point did your whole group disagree with and why?
- About which talking point were you most unsure and why?
- Which talking point do you know you are right about and why?
- Could any of the talking points be true and false?

Source: https://kgmathminds.com/2017/02/05/fraction-talking-points-3rd-grade/

"As a class, we reviewed the process and practiced Talking Point #1 together as a class. From there I let them go and circulated the class to hear the conversations! It was the absolute highlight of my first week!" (Kristin Gray, source below)

Source: https://kgmathminds.com/2014/09/06/week-one-talking-points-math-mindset/

UNIT 4: GENDER AND INCLUSIVENESS IN MATHEMATICS EDUCATION

Introduction

Areas with consistent gender differences are children's **beliefs about their abilities in mathematics**, **their interest in mathematics** and their **perceptions of the importance of mathematics for their future**. In all three domains, girls have found to be scoring lower than boys. Researchers have found that girls often have less confidence in their mathematics abilities (Zuze & Lee, 2007). This is a problem because research shows that children's beliefs about their abilities are central to determining their interest and performance in different subjects and the career choices they make (Beilock, Gunderson, Ramirez, & Levine, 2010).

These gender differences contrast with research that males and females generally show similar abilities in mathematics (Hyde, Fennema, & Lamon, 1990). In the case of Rwanda, data show that both girls and boys face gender-related barriers to learning. In the national examination results, boys outperformed girls in almost all districts at P6 and S3 levels during the period 2008-2014 (MINEDUC, 2015).

An analysis of data shows the percentage of children making it from P1 to P6 in the previous six years was only 10% on average; for boys, the percentage was slightly lower than for girls (NISR, 2015). This shows that, while girls face many challenges related to learning, progression and completion, also boys face challenges that include repeating and dropping out of primary school (NISR, 2015).

To eliminate all the causes and obstacles which can lead to inequity in education, the Ministry of Education included gender as one of the crosscutting issues in the pre/primary and secondary Competence Based Curriculum framework (Rwanda Education Board, 2015).

This section aims therefore at equipping mathematics teachers with the competences to apply a gender responsive pedagogy in their teaching.

Learning Outcomes

By the end of this unit, participants should be able to:

- Understand the meaning of gender and inclusive education;
- Address crosscutting issues with focus on inclusiveness and gender in teaching and learning mathematics;
- Apply a gender responsive pedagogy in the classroom;
- Design learning activities that will equally interest and engage all learners in mathematics;
- Support fellow teachers in applying gender responsive pedagogy and inclusive education;
- Make learning of mathematics enjoyable for all learners;
- Acknowledge the presence of gender stereotypes in mathematics education;
- Appreciate that boys and girls have on average equal abilities to achieve proficiency in mathematics;
- Commit to working towards gender equity and inclusiveness in their school;
- Respect for the diversity in learners' feelings, opinions and abilities.

Section 1: Gender

What Is Gender?

Activity 26

In groups of 4, create a story to tell:

1. A lion………
2. A zebra……….
3. Once upon a time a girl………
4. Once upon a time a boy………

You can be as creative as possible.

Gender is a concept that is widely used and perceived by many to mean "women's issues". In reality, gender refers to the socially determined roles and relations between males and females (Subrahmanian, 2005). Gender is different from sex. Sex refers to purely biological differences between men and women. Gender roles, on the other hand, are created and sustained by the society, which assigns different responsibilities to men and women, e.g., cooking for women and decision-making for men.

Gender roles can therefore be changed and vary over time and from community to community. These gender roles are consciously or unconsciously carried into the classroom by teachers, students, school leaders, parents and other stakeholders. In children's textbooks, for example, women are often represented as cleaners, caregivers and nurses, and men are drivers, doctors and leaders. The images reinforce gender roles, which are socially constructed.

Activity 27

What is the influence of cultural norms and practices on girls' participation in mathematics classes in your school?

Key Terms

Several related concepts underlie the development of a clear understanding of gender:

Gender discrimination: Denying opportunities and rights or giving preferential treatment to individuals based on their sex. For example, only giving boys the opportunity to be a team leader.

Gender equality: The elimination of all forms of discrimination based on gender so that girls and women, boys and men have equal opportunities and benefits (OECD, 2015). For example, giving an equal chance to boys and girls to be a team leader.

Gender equity: Fairness in the way boys and girls, women and men are treated. In the provision of education, it refers to ensuring that girls and boys have equal access to enrolment and other educational opportunities (Subrahmanian, 2005). For example, giving additional support to girls so they can become confident to volunteer for team leader.

Gender stereotype: The constant presentation, such as in the media, conversation, jokes or books, of women and men occupying social roles according to a traditional gender role or division of labour (OECD, 2015). For example, a textbook where always boys names are used to describe team leaders.

Gender sensitive: The ability to recognize gender issues. It is the beginning of gender awareness (UNICEF, 2017). For example, a teacher who is aware that boys are always team leaders and that something should be done about this.

Gender parity: This refers to the equal representation of boys and girls (UNICEF, 2017). For example, in a class, there is an equal number of male and female team leaders.

Figure 24 illustrates the difference between equality and equity. Equity is about giving each learner the support he or she needs to achieve the learning outcomes. This implies that some learners will need more or different support than others.

Figure 24: Equality versus Equity (Save the Children, Mureke Dusome project, 2017)

Gender Responsive Pedagogy

Introduction

Observations of classroom practices show that teaching and learning is often gender biased (Aikman & Underhalter, 2007; Consuegra, 2015). Many teachers apply teaching methodologies that do not give girls and boys equal opportunities to participate and learn. They also use teaching and learning materials that perpetuate gender stereotypes. Therefore, it is important for teachers to apply a gender responsive pedagogy.

Gender responsive pedagogy means that teaching and learning processes pay attention to the specific learning needs of girls and boys (Mlama, 2005). It does not mean treating boys and girls equally. It includes lesson planning, teaching, classroom management and evaluation.

In this section, we discuss some strategies that teachers can use to promote the involvement and learning of girls in mathematics lessons.

Keep in mind thought that many techniques that we discuss in this course aim at involving all learners. None of these strategies, however, is automatically gender responsive. Often, boys dominate learning processes in the class. Therefore, teachers need to consider the specific gender needs of girls and boys in planning their lessons. Being gender responsive does not means treating all learners equally but making sure that all learners have equal opportunities to learn.

Activity 28

Think individually about what teaching approaches you have used to encourage equal participation and achievement of boys and girls in your lessons. Afterwards, discuss your ideas with your neighbour. Next, the facilitator will organize a plenary discussion.

In their lesson plans, teachers should consider how all learners will participate in learning activities. They should ensure that there is equal participation in activities such as making presentations, conversations and practical activities. In group activities, ensure that girls and boys are given leadership positions and roles. Consider how learning materials will be distributed equally to both girls and boys, especially in case of shortages.

Things that you can do to make classes gender equitable

1. **Using gender neutral language**

Gender responsive pedagogy includes **gender neutral language use** by the teacher. Teachers often discourage girls from doing mathematics by telling them that such subjects are for boys or too difficult for girls. When a girl is assertive, she is told to stop behaving like a boy, and when a boy cries, he is cautioned to stop behaving like a woman.

Much gender insensitive communication is **non-verbal**. An indifferent shrug of the shoulders or rolling of the eyes suggests that the student is too foolish or annoying to deserve attention. Other gestures and body language, such as winking, touching, brushing, grabbing and other moves may be overtly sexual. This type of communication may go unnoticed by others for a long time, but it can be very damaging to classroom participation for the learner at whom the communication is targeted.

2. **Classroom arrangement**

Since girls are not brought up to speak out – or rather, are brought up not to speak out – when they sit at the back of the class, they are less likely to participate unless the teacher makes a special effort to involve them. Remember the distinction between equality and equity. Breaking the class into smaller groups may encourage girls to participate.

3. Teach learners that learning abilities are improvable

To enhance girls' beliefs about their abilities, teachers should understand and communicate this understanding to students:

Mathematics abilities can be improved through consistent effort and learning. Research shows that even students with high ability who view their cognitive skills as fixed are more likely to experience discouragement, lower performance and reduce their effort when they encounter difficulties. Such responses are more likely in the context of mathematics, given stereotypes about girls' mathematics abilities (Dweck, 2006). Negative stereotypes can lead girls to choose unchallenging problems to solve, lower their performance expectations and not consider mathematics as a career choice.

In contrast, students who view their abilities as improvable tend to keep trying in the face of difficulty and frustration to increase their performance.

More information: https://www.youtube.com/watch?v=fC9da6eqaqg

4. Expose girls to female role models

Girls who only encounter men as maths and science teachers may be confirmed in their beliefs that mathematics and science are for men. If there are no female mathematics and science teachers in your school, you can still introduce them to examples of women who achieved a lot in mathematics and science.

Teachers can invite women or older students as guest speakers or tutors. These role models should communicate that becoming good at mathematics takes hard work and that self-doubts is a normal part of the process of becoming an expert.

Activity 29

Joining the ranks of Neurosurgery: My Impossible Dream | Claire Karekezi.

After 15 years of intense training and studying that has taken her across three continents, Dr Claire Karekezi returns home to Rwanda as the only female neurosurgeon in the country.

Video: https://www.youtube.com/watch?v=s2dD2w_4vso

Discussion questions:

1. Why is it important to expose girls to women who achieved a lot in mathematics?
2. How can female role models help with achieving gender equity in mathematics in your school?

Example of **role models for mathematics and science in Africa**: **Apps and Girls**

Apps & Girls is a Tanzanian registered social enterprise that was founded in July 2013 by Carolyne Ekyarisiima. It seeks to bridge the tech gender gap by providing quality coding training (web programming, mobile app development game development and robotics) and entrepreneurship skills to girls in secondary schools via coding clubs and other initiatives such as mentorships and scholarships. So far, they have created 25 coding clubs in Tanzania and they have trained 269 teachers and 2656 girls. They want to train 1 million girls before 2025.

Link website: http://www.appsandgirls.com/

Link YouTube: https://www.youtube.com/watch?v=yNNrVqUvkjg

5. Gender-responsiveness in classroom interactions

Many techniques that we have discussed in this guide aim at improving the quality of interactions in the mathematics classroom, both between teacher and learners and between learners. In managing these interactions, it is important as a mathematics teacher that you are aware of potential gender bias and that you can act to address this. In Table 6 we list some guidelines to ensure that conversations and group activities are gender responsive.

Table 6: Actions to make classroom interactions more gender responsive (Mlama, 2005)

Methodology	Action
Conversations (questions and answers)	▪ Give equal chances to both girls and boys to answer questions, including more difficult questions. ▪ Give positive reinforcement to both girls and boys. ▪ Allow sufficient time for students to answer questions, especially girls who may be shy or afraid to speak out. ▪ Assign exercises that encourage students, especially girls, to speak out. ▪ Distribute questions to all the class and ensure that each student participates. ▪ Phrase questions to reflect gender representation – use names of both men and women, use both male and female characters.
Group activities	▪ Ensure that groups are mixed (both boys and girls). ▪ Ensure that everyone has an opportunity to talk and lead the discussion. ▪ Ensure that group leaders are both boys and girls. ▪ Encourage both girls and boys to present results. ▪ Ensure that both girls and boys record outcomes.

Activity 30

Have you tried one of the approaches discussed above in Table 6? If so, what have been your experiences? If no, is there anything that prevents you from trying them out?

Section 2: Inclusive Education

What is Inclusive Education?

Inclusive education is based on the idea that **all learners are different but have the capacity to achieve the learning outcomes**. Inclusive education means adapting teaching to meet the needs of each individual learner.

The CBC identifies special needs as a cross-cutting issue in all subjects. Therefore, teachers are called to identify students who are struggling mathematically and adjust the learning environment to enable them to learn. Inclusive education is about treating all learners as individuals. It is about making sure that all learners can learn. Therefore, inclusive education is **much broader than special needs education**, which focuses on learners with disabilities.

Activity 31

Describe in one sentence what inclusive education means to you practically in your daily teaching. Compare and discuss with your neighbour. Try to come to an agreement.

When we think about inclusive education, often we think about getting children into school, i.e. making sure they are present in school. However, we also need to ensure that children are participating in lessons and school life, and that they are achieving academically and socially as a result of coming to school.

When thinking about inclusive education, always consider **Presence, Participation and Achievement** (Figure 25) (Ainscow, 2005).

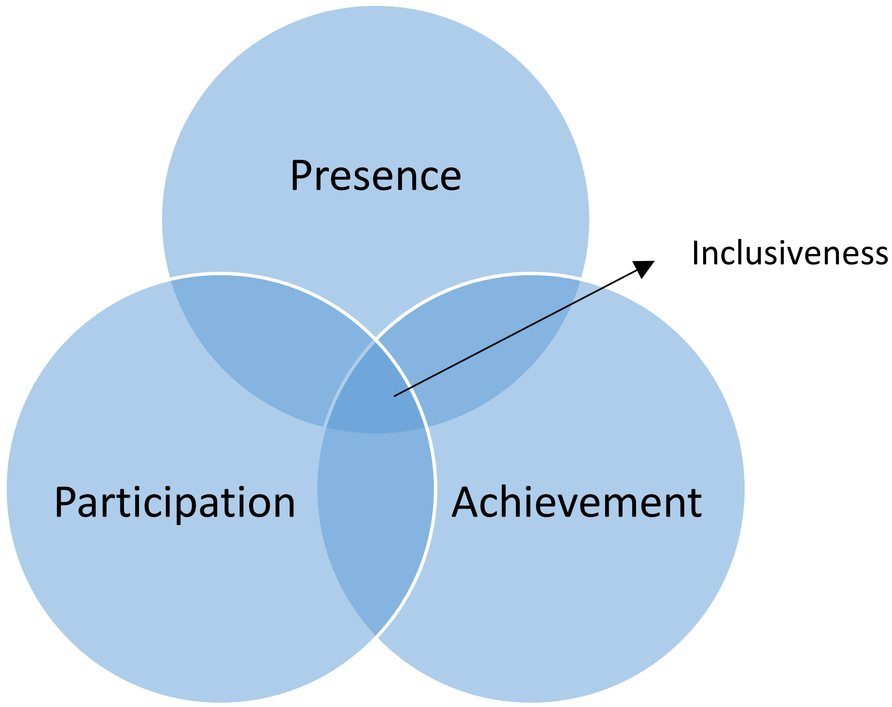


Figure 25: Components of Inclusive Education (Ainscow, 2005)

It is not enough that they simply attend the lessons; all children should be given the same opportunities to fully participate and achieve.

Equal Presence: Teachers should be instructed to do daily attendance of all children. If there is an attendance issue related to sex, disability or other reason, talk with parents through School General Assembly meetings. Invite the concerned parents at school to speak about why all learners should be provided with equal learning opportunities and how to support their learning needs.

Equal participation: Teachers should ensure that all learners are participating actively and given chances to lead in classroom activities, classroom discussions, and different clubs.

Equal achievement: Parents, teachers and school leadership should ensure that all learners have equal opportunities to access learning materials and that there are not any systematic achievement gaps. You may think it is too difficult to address the needs of a diverse range of children, as there are so many challenges. However, by working as a team within your school, with support from families and local communities, and by making small changes to your teaching methods, you will be able to meet the needs of all children.

Differentiation

Differentiation is a key classroom strategy to make teaching and learning more inclusive. But what does differentiation really mean? Is it feasible in classrooms with many learners and how do you go about it?

Activity 32

Discuss the following statements (agree/not agree/ depends)

1. Differentiation is an idea as old as effective teaching
2. Differentiation means grouping students by ability
3. Differentiation is mostly aimed at students with identified learning challenges
4. Differentiation is about valuing and planning for diversity
5. Differentiation means that all students do different things
6. Differentiation is not possible in classes with more than 50 learners.

You can find the solutions in the infographic below.

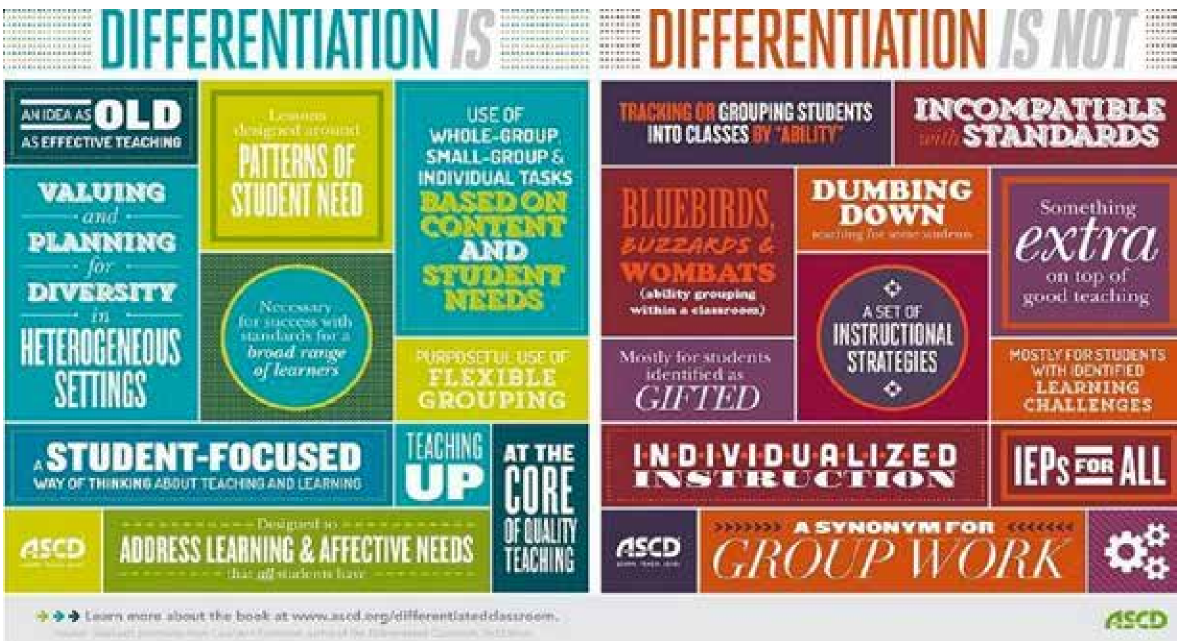


Figure 26: Differentiation is & Differentiation is not (ASCD, 2015)

Source: http://www.ascd.org/ASCD/pdf/siteASCD/publications/Differentiation_Is-IsNot_infographic.pdf

Every classroom at every grade level contains a range of students with varying abilities and backgrounds. In Rwanda, many students' mastery of learning is several grade levels below the grade they are in. Perhaps the most important work of teachers is to identify students' level of prior knowledge in mathematics and then plan lessons that support and challenge all students to learn. This will enable teachers to differentiate instruction effectively through, considering the large class sizes, providing remediation for struggling learners so they can catch up with the rest of their peers.

Activity 39

A mathematics teacher teaches in grade 5, but notices that some students do not have the required prior knowledge on fractions that they should have learned in grades 3 and 4.

What would you do?

Differentiation is not about having learners do different things all the time, nor is it about teachers choosing the learning for them, it is **about learners doing the same thing in different ways**.

How to differentiate

A first step in differentiating teaching is **taking the knowledge that learners bring to class into account**. There is evidence that learning is improved when teachers pay attention to the prior knowledge and beliefs of learners, use this knowledge as a starting point for teaching and monitor learners' changing conceptions as the lesson proceeds. If their initial understanding is not engaged, they will fail to understand the new content. There are four approaches to differentiation (Figure 27):

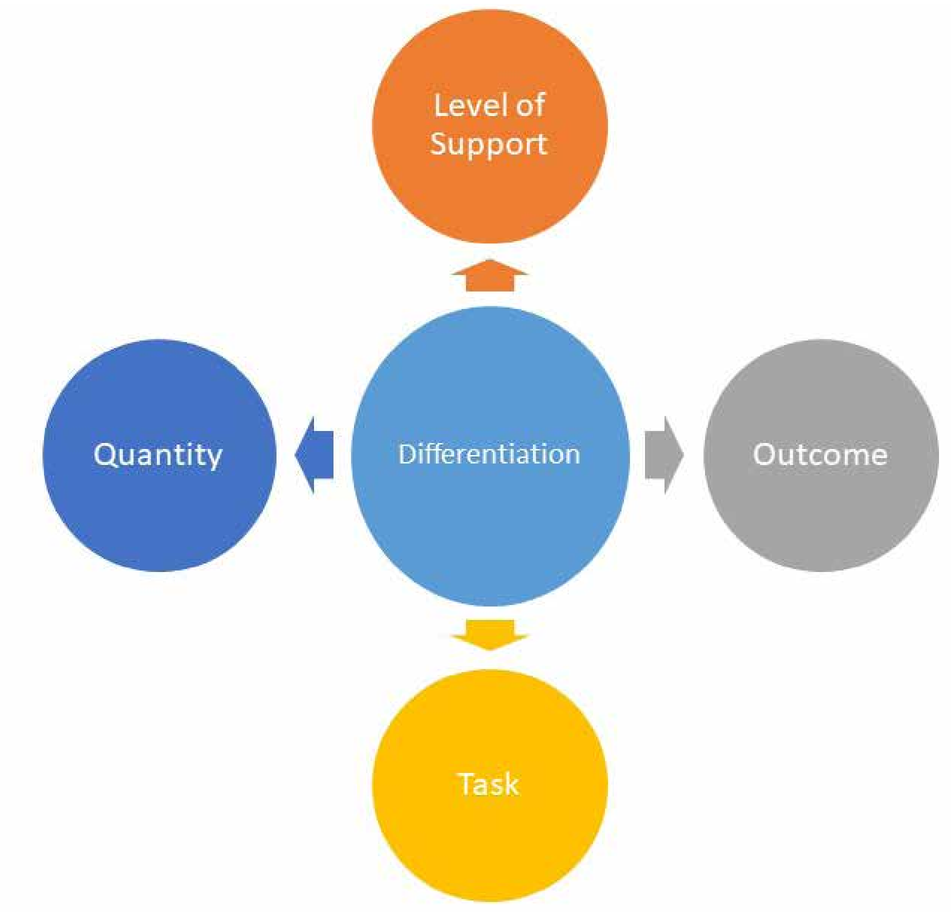


Figure 27: Approaches to differentiation

Differentiate by quantity

You can give some learners 8 questions and other learners 12 questions. This approach assumes that better performing learners work faster, and extra work should be prepared to cater for this. However, 'more work' is unhelpful when it only means 'more of the same'. These learners need to explore ideas in more depth, not merely cover more content.

Differentiate by task

In this approach, learners are given different problems or activities, according to their learning needs. This approach is difficult to implement well, because it assumes that the teacher can judge the performance of each learner accurately and that there is also a supply of suitable problems or activities. If you decide in advance that some learners will not be able to cope with particular concepts and ideas, you deny them the opportunity to engage with these ideas. It is therefore not a good strategy to simplify activities for some learners in advance.

A better approach is to **give learners some choice in the activities they undertake**. For example, learners can be asked to choose between an easy, a challenging and a very challenging task. Research showed that few learners choose the easy task and that most prefer a challenge (Swan, 2005). Figure 28 shows an example of differentiation by task for a lesson about perimeters (P3).

An example of good practice in a Rwandan school:

A P3 Mathematics teacher in Kayonza district modelled this very well during a lesson on perimeter. Rather than only providing one or two questions for pupils to work on independently, he provided 4 questions. The questions were of varying levels of difficulty, allowing **all** pupils to achieve, from the pupils with learning difficulties to those who were able to work at a faster pace.

These are the questions that the teacher in Kayonza created and used:

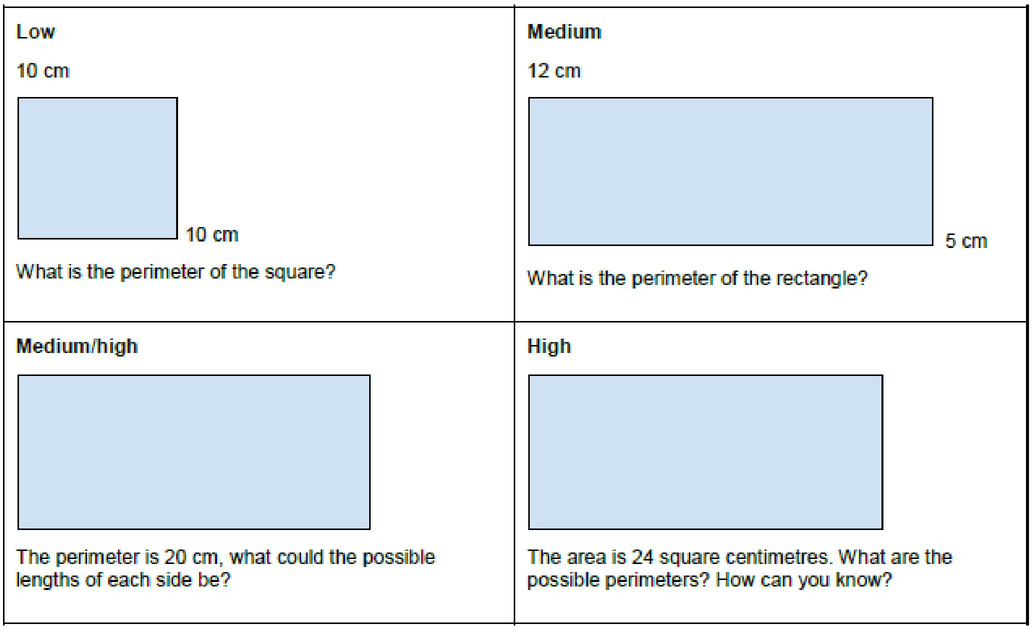


Figure 28: Example of differentiation by task for P3 mathematics (BLF, 2019)

Differentiate by level of support

In this approach, all learners are given the same task, but are offered different levels of support, depending on the needs that arise during the activity. This avoids the danger of prejudging learners. For example, you may give carefully chosen hints during a group work activity. In a task about positive and negative integers, numbers lines could be provided for those that need them.

Differentiate by outcome

Open activities that encourage a variety of possible outcomes offer learners the opportunity to set themselves appropriate challenges. This approach is used in many of the activities in this guide. For example, some activities invite learners to create their own classifications or their own problems and examples. Teachers may encourage learners to 'make up questions that are difficult, but that you know you can get right'.

Activity 33

Review the four ways to differentiate teaching. Give an example from your mathematics teaching for each of them. Which one do you use the most? Which one the least? Why?

Strategies to implement differentiation in mathematics

1. **Identify and focus on key concepts**

Determine for each lesson what the key concepts are that each learner should master at the end of the lesson. The CBC provides you with a starting point for what the key concepts are.

For example, a Grade 6 teacher is planning a lesson on multiplying whole numbers by decimals. Although the goal of the instruction is performing a computation like 1.5 x 3, the key concept that students need to understand is that multiplication has many meanings (e.g., repeated addition, counting of equal groups, objects in an array, area of a rectangle).

2. **Designing Open Tasks**

Suppose a P4 teacher wants to teach the key concept that any subtraction can be thought of in terms of a related addition. P4 students should be able to solve addition and subtraction problems involving multi-digit numbers, using concrete materials and standard algorithms, as well as use estimation to help judge the reasonableness of a solution. Some students may not be ready to deal with three-digit numbers, even with the use of concrete materials. A teacher might change the planned task to turn it into an open task (Figure 29).

Open tasks are also called "**Low threshold, high ceiling tasks**". The low threshold means that the task can be done by learners who still have a low understanding of the concept. The high ceiling means that the task can still be challenging for learners who have already a good understanding of the concept.

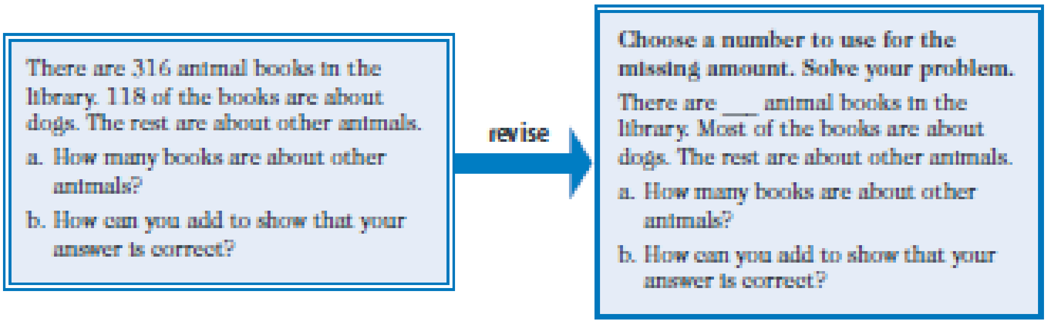
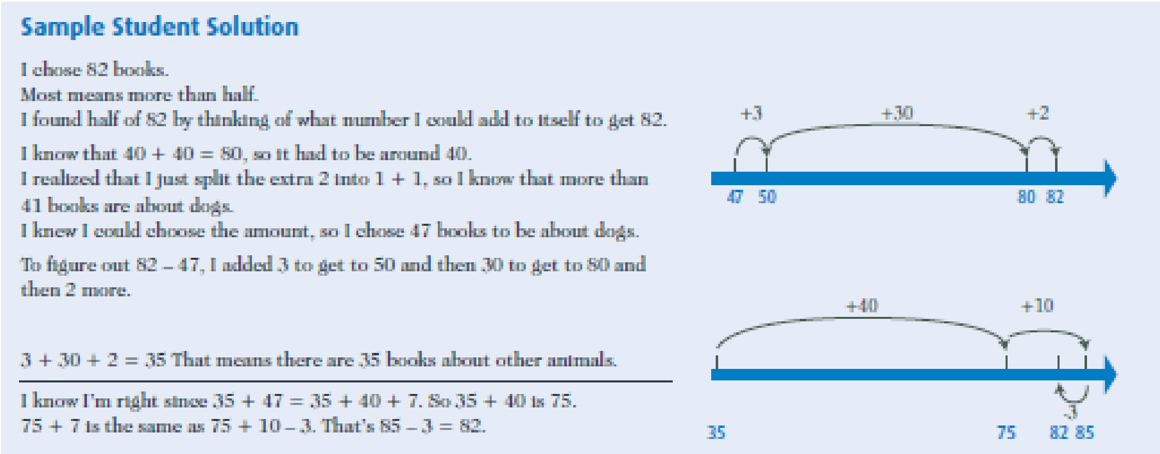


Figure 29: Example of Open Task (Beckmann, 2013)

With the open number task, students have a choice in the numbers they use, choice in the strategies they use and a choice in how they interpret the meaning of the problem. Students who can only handle numbers below 20 can do so. Students who can handle numbers below 100 in a concrete way can do so too. Students who are ready to work with very large numbers can do so as well. Also, in the revised task, some students will interpret the phrase "most of the books" to mean more than half. Others can simply interpret it as meaning that more books are about dogs than other animals; they might make a list of different animals with a total number of books about each animal, ensuring that the number for dogs is the greatest number on the list. These variations really don't matter. All students will be considering a subtraction situation; all of them are relating it to an addition situation; all of them have an opportunity to understand and solve the problem using their own student-generated strategies and appropriate materials. Whether students are working with large or small numbers, the sharing of their mathematical thinking is valuable for the collective learning of the class.



Sample Student Solution

I chose 82 books.
Most means more than half.
I found half of 82 by thinking of what number I could add to itself to get 82.
I know that 40 + 40 = 80, so it had to be around 40.
I realized that I just split the extra 2 into 1 + 1, so I know that more than 41 books are about dogs.
I knew I could choose the amount, so I chose 47 books to be about dogs.
To figure out 82 – 47, I added 3 to get to 50 and then 30 to get to 80 and then 2 more.

3 + 30 + 2 = 35 That means there are 35 books about other animals.

I know I'm right since 35 + 47 = 35 + 40 + 7. So 35 + 40 is 75.
75 + 7 is the same as 75 + 10 – 3. That's 85 – 3 = 82.

Figure 30: Example Solution for Open Task (Beckmann, 2013)

In fact, there might be more mathematically sophisticated thinking from a student who uses a smaller value than one who simply uses a standard algorithm to subtract 118 from 316. With several differentiated student responses to the problem, it is valuable for students to share their thinking and compare strategies. In this example, the teacher can co-ordinate a class discussion about the use of different models of representations to show different mathematical thinking:

- Some students might use an empty number line. This has the benefit of flexibility; students can use numbers in whatever increments make sense to them.
- Other students might use base ten blocks and focus on place value concepts. These students practise the important skill of decomposing numbers into their hundreds, tens and ones (units) components.
- Some students might draw diagrams. For example, the student might draw a model for 316 – 118. The model reinforces the mental concept that to subtract 118 from 316, you can think of subtracting 116 and then another 2, to get 316 – 116 = 200 and 200 – 2 = 198.

In the example below, children can choose various combinations of numbers to solve the problem.

There were ________ children on the playground.
________ more came to join them.
How many children were on the playground then?

Figure 31: Example of open learning task

Another example: Sarah and Mike ran each day this week. Each day Sarah ran 3 kilometres in 30 minutes. Mike ran 6 kilometres in 72 minutes. Here are the answers: 42, 2, 294, 3 ½. What can be the questions for each answer?

Possible responses:

42: How many more minutes did Mike run than Sarah each day?

2: How many more minutes does it take Mike to run a kilometre?

294: How many more minutes did Mike run this week than Sarah?

3,5: How many hours did Sarah run this week?

Example: Practicing differentiation with a card sorting activity

In this activity, learners classify information written on cards according to whether it describes a linear or non-linear equation and then into four given sub-categories.

Card sorts can be adapted to suit the group or level of ability. In an easy card sort you can classify cards into two simple categories such as linear or non–linear. You can also differentiate by leaving some cards out for less able learners or by **adding more or harder cards to challenge the stronger learners**.

You may include some **blank cards** for learners to write their own equation for each of the types. Learners can then be given the incentive to produce a difficult example by telling them their card will be used by other groups and in other classes. This is an example of an open-ended task.

Linear	*Non–linear*
Don't know	

If you include a 'don't know' box you can talk to learners about the cards they have put in the 'don't know' box

Linear			Non-linear
Horizontal lines	Positive gradient	Negative gradient	
$y = -3$ $y = 0.6$ $y = 2$ $0 = y$	$y = \frac{3x}{5}$ $t = 3p$ $y = 3x - 6$ $y - 9 = 7x$ $y + 3 = 4x$ $y = x$ $2y - 4x = 7$ $x = 2 + y$ $m = 7 + 3p$ $c = 8p + 2$ $r = t$	$y = 2 - 4x$ $y - 1 = -x$ $y = \frac{-x}{5}$ $5y - 8 = -3x$ $2y - 7 + 4x = 0$ $y + 3x = 9$ $3x = 5 - y$ $x + y = 0$	$x^2 + y^2 = 8$ $x = \frac{1}{y}$ $y = x^2 + 8$ $y = x^2 + 3x - 7$ $x^2 = y - 1$ $y = x^3 + 2x^2 - x$ $y = \frac{2}{x}$
Vertical lines			
$x = -1$ $x = {}^1/_3$ $x = 0$ $7 = x$			

3. Regularly checking students' understanding

Structure your lesson in such a way that there are frequent moments for checking learner understanding. Avoid long series of exercises where students may get stuck for a long time. Some struggle is fine for students, and even helps learning and retention, but avoid that they get completely stuck and become demotivated.

For example, learners make a few exercises. When they have finished, they raise their hand for a quick check. If ok, they can proceed with the next exercises. If the same errors keep coming back, you can build in a moment of whole-class instruction. This technique allows for accommodating both faster and slower learners.

4. **Involving learners with disabilities**

Differentiation does not require the specialized knowledge to deal with specific learning disabilities. However, as a teacher you can take some simple steps to help learners with learning difficulties. Table 7 lists some classroom strategies to help learners with various learning challenges.

Table 7: Learning Challenges and Possible Classroom Strategies (Save the Children, Mureke Dusome project, 2017)

CHALLENGE	CLASSROOM STRATEGY TO ADDRESS
HEARING	▪ Try to convey information to the child using sign language or informal signs and hand gestures. ▪ Seat the child in the front row. Speak loudly and clearly. ▪ Ensure the child can see your mouth when you speak. ▪ Provide the child with a detailed outline of the lesson/objectives. ▪ Use charts, pictures and icons to convey information. ▪ Assign the child a learning buddy. ▪ Speak with the child's parents to identify and build on communication techniques used at home.

CHALLENGE	CLASSROOM STRATEGY TO ADDRESS
TALKING	▪ Encourage the child to continue when he/she is trying to communicate. ▪ Be attentive while he/she is talking. ▪ Provide opportunities to use different ways of communication such as role play, gestures, drawing, writing, etc. ▪ Speak with the child's parents to identify and build on communication techniques used at home.
PHYSICAL ACCESS	▪ Ensure the child is physically able to access his/ her classroom and seat. ▪ Ensure the child can access learning materials. ▪ Assign a student helper or circle of friends to help the child navigate the classroom.
READING	▪ Ask the child to follow along with a finger. ▪ Provide a piece of paper or other material and instruct the child to uncover one sentence at a time while reading. ▪ Provide extra reading practice time in school and at home. ▪ Pair the child with a reading buddy who reads with him/her daily.
SEEING	▪ Ensure that the classroom has good lighting. ▪ Write in large clear letters on the blackboard. ▪ Assign the child a learning buddy. ▪ Seat the child in the front row. ▪ Refer the child for glasses, if possible.

Activity 34

Review Table 7. Can you give examples strategies that you will try out in your school? Do you use other strategies to help learners with specific learning disabilities? Discuss your experiences and ideas in small groups. If time permits, groups can briefly present and discuss their findings.

UNIT 5: ASSESSMENT

Introduction

Assessment is a crucial element in teaching and learning (Hattie, 2009). Quality assessment provides information to students, teachers, parents and the education system in effective and useful ways. To be helpful, however, it must be broad ranging, collecting a variety of information using a range of tasks before, during and after a teaching sequence. Assessment is more than the task of collecting data about students' learning. It includes the process of interpreting the collected data and acting upon those judgements during teaching. There are different types of assessment like diagnostic, formative, summative assessment, etc.

Learning Outcomes

By the end of this unit, participants will be able to:

- Explain principles of formative and summative assessment in the competence-based approach;
- Understand the role of formative assessment in improving learners’ performances;
- Design tools for formative and summative assessment;
- Conduct formative and summative assessment with the objective to improve learner’s performance;
- Support fellow teachers to organise formative and summative assessment activities and use data from the assessment to improve learners’ performance;
- Appreciate the role of assessment within quality mathematics teaching and learning.

Section 1: Formative Assessment

What is Formative Assessment?

The goal of formative assessment is to monitor student learning frequently to provide feedback for teachers to improve their teaching and for students to improve their learning (Black & Wiliam, 2001). More specifically, formative assessment:

- helps students identify their strengths and weaknesses and target areas that need attention;
- helps teachers recognize where students are struggling and address problems immediately;
- enables teachers to build on learners' prior knowledge, and match their teaching to the needs of each learner

Formative assessment should only have a low impact on learners' final grades. Examples of formative assessment include asking learners to:

- complete a short quiz at the start or end of the lesson;
- write short notes summarizing the main ideas of the lesson;
- work in groups to make a poster or presentation on a topic.
- use voting cards to answer the teacher's questions

Conducting Formative Assessment

Activity 35

Write down on a flashcard three formative assessment activities that you do during your teaching.

Post the flashcards on the wall.

Activity 36

Which techniques from unit 3 can you use to conduct formative assessment in your class? After a few minutes of individual reflection, share your ideas with your neighbour.

Discussion

Many aspects of mathematics teaching that we discussed in unit 3 can be used for formative assessment. Effective questioning gives you information about learners' level of understanding. Their ability to reason and engage in mathematically correct conversations shows you whether they have mastered the concepts of the lesson. The level of problems that learners can solve and formulate shows you their level of understanding. Using techniques for addressing misconceptions such as concept cartoons helps you to assess whether frequently occurring misconceptions are present among learners. Learners' ability to move between concrete, pictorial and abstract representations of concepts is a great indicator of their understanding. Finally, observing interactions among learners during games and group work activities will give you valuable information on learners' level of understanding. Therefore, all key aspects that we discussed in Unit 3 can be used for formative assessment. Below, we highlight some specific techniques for formative assessment.

Techniques for formative assessment

1. **Share learning objectives with learners**

Formative assessment involves both the teacher and the learners. Therefore, the first step is that learners know what the learning objectives of the lesson are. Often, the teacher knows why the learners are engaged in an activity, but learners cannot always differentiate between the activity and the learning that it is meant to promote. Explicitly sharing the learning objectives will direct learners' attention to the learning. The learning objective is expressed in terms of knowledge, understanding and skills, and links directly with the CBC.

The design of learning objectives starts with the answers to these questions.

- What do I want students to know?
- What do I want students to understand?
- What do I want students to be able to do?

When students know the learning objectives of a lesson, they are helped to focus on the purpose of the activity, rather than simply completing the activity.

2. Plan assessment opportunities during lessons

Researchers recommend using small, frequent tests with good feedback. It is the feedback on what they don't know, not that which the student got right, that leads to learning (Black & William, 1998). As well as informing teachers, planned assessment should also help learners become more aware of what they still need to learn and how they might go about learning these things.

Research in Rwanda found that there is little or no time to gather, analyse and use assessment information to improve learning and inform planning. This prevents teachers' ability to get to know their learners personally, differentiate appropriately, as well as improve the effectiveness of teaching

3. Encourage self-assessment and peer-assessment

Studies on formative assessment point to the value of learners assessing themselves. Through this process learners become aware of what they need to know, what they do know, and what needs to be done to narrow the gap. One way of achieving this is to give copies of learning objectives to learners, ask them to produce evidence that they can achieve these objectives and, where they cannot, discuss what they need to do next. Over time, it is also possible to foster a collaborative culture in which learners take some responsibility for the learning of their peers. This involves making time for learners to read through each other's work and to comment on how it may be improved.

4. Give feedback that is useful to learners

Evidence suggests that the only type of feedback that promotes learning is a meaningful comment (not a numerical score) on the quality of the work and constructive advice on how it should be improved (Nicol, 2007). Indeed, grades usually detract learners from paying attention to qualitative advice.

The research evidence (Black & Wiliam, 2001; Nicol, 2007; Hattie & Timperley, 2007) clearly shows that helpful feedback:

- focuses on the task, not on grades;
- is detailed rather than general;
- explains why something is right or wrong;
- is related to objectives;
- makes clear what has been achieved and what has not;
- suggests what the learner may do next;
- describes strategies for improvement.

This doesn't necessarily mean writing long comments at the bottom of each piece of work. It is helpful to give comments orally and then perhaps ask learners to summarise what has been said in writing.

5. "Exit ticket" Technique

An exit ticket is a brief evaluation that students write and turn in before the end of the class (Lemov, 2015). It should have only 2 or 3 short questions or problems and show what they have remembered from the day's class. This can provide valuable information on who learned what and who needs more help. It can help you respond to individual students' needs and decide on what to focus in the next lesson. It is a kind of formative assessment that informs the teacher, but also the learners about how well they have understood the key outcomes of the lesson.

Good exit tickets:

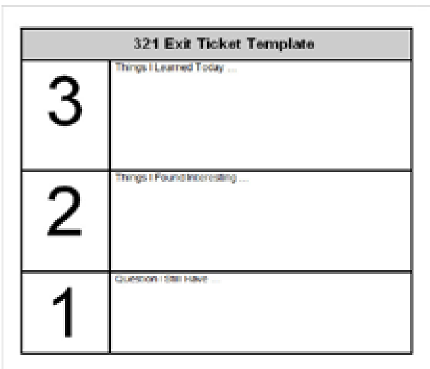
- Contain just a few questions.
- Contain questions of different types (e.g., one multiple choice, one open-ended question)
- Answers can be analysed quickly by the teacher.
- Questions relate to the key objective(s) of the lesson.

Questions that encourage student self-reflection can also be used, possibly in combination with content-oriented questions:

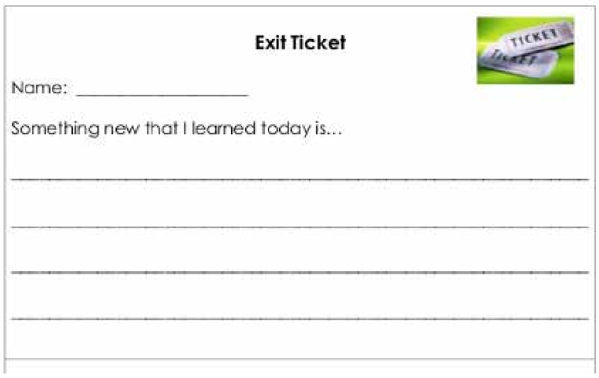
- What did you find the most important idea of the lesson?
- What did you find difficult and would like more exercises or explanation on?
- How does the lesson relate to what you have learned before?
- Write one question you still have

Table 8 shows two possible templates for an exit ticket. The main idea is that it is short and allows you to get a quick insight in students´ mastery of the key outcomes of the lesson.

Table 8: Templates for Exit Tickets


321 Exit Ticket Template

321 Exit Ticket Template	
3	Things I Learned Today ...
2	Things I Found Interesting ...
1	Question I Still Have ...


Exit Ticket

Name: ________________

Something new that I learned today is...

An exit ticket should give you quick data. It is important that you follow up on the results of the exit ticket. If most students have a problem with the first question, look at the kinds of problems students encountered, and model the way to correct the problem (Lemov, 2015). You select some common mistakes for discussion the next day or you can put some students in a separate group for remedial instruction or exercises.

More information on exit tickets: https://buildingmathematicians.wordpress.com/2016/07/04/exit-cards-what-do-yours-look-like/

6. Using "Traffic Light Cards" and "Voting Cards"

Traffic light cards and voting cards are cards that are used by learners to respond to questions from the teacher (Figure 32).

Traffic light cards are used by learners to communicate their understanding about a topic:

1. Raising a red card means: "I'm stuck, I need some extra help"
2. Raising an orange cards means: "I'm not quite sure, I need a little help"
3. Raising a green card means: "I fully understand, I don't need any help"

A teacher can use the technique at the end of parts within a lesson. A lot of red cards mean that many learners are still struggling. It shows the need for additional instruction or more exercises. A situation with few yellow or red cards shows the teacher that some learners do still have problems. They may be taken apart by the teacher for additional explanation. If there are some learners with green cards, the teacher may ask them to explain the concept to those with red or yellow cards.

Learners may vote not according to what they think, but what others do. Therefore, it is good to follow up the voting with a few questions like: "Emile, you voted red, what is it that you find difficult?", or, "Emmanuel, you voted green, can you explain the key idea to the others?"

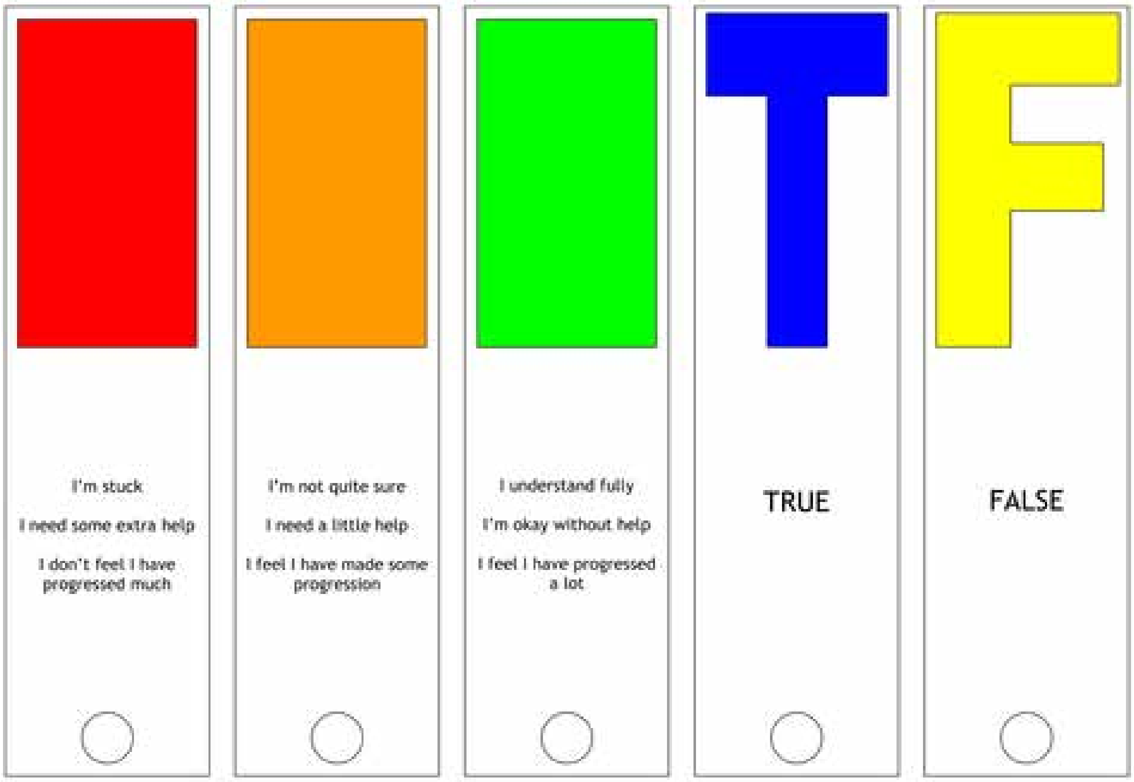


Figure 32: Traffic Light Cards and Voting Cards (TES, 2013)

Voting cards are used by learners to vote for a specific answer on a question by a teacher. This can be a true-false question (Figure 32) or a multiple-choice question (Figure 33).

Figure 33: Voting cards with letters

You can print these cards for your learners. If possible, laminate them, so they will keep longer. You can combine colours and letters on the front and back side.

Section 2: Summative Assessment

What is Summative Assessment?

Activity 37

Write on a flashcard your understanding of summative assessment and give two examples. Post your flashcard on the wall.

Bloom, Hastings, & Madaus (1971) define summative evaluation as assessment given at the end of units, mid-term or at the end of a course, and which is designed to judge the extent of students' learning of the material in a course, with the purpose of grading, certification, evaluation of progress or even for researching the effectiveness of a curriculum. The goal of summative assessment is to evaluate student learning at the end of a unit or term by comparing it against standards or outcomes (Black and Wiliam, 2001).

Examples of summative assessment include:

- a midterm exam
- P6 national examination
- a final project or portfolio of evidence

Activity 38

Critically analyse a sample of a PLE paper and a sample of teacher's exam paper.

Differences between Summative and Formative Assessment

Unlike formative assessment, summative assessment is not part of the instructional process. Summative assessments happen too far down the learning path to provide information at the classroom level and to adjust and intervene during the learning process.

However, formative and summative assessment are connected (Figure 34). Information from summative assessment can be used formatively when students or teachers use it to guide their efforts and activities in their teaching.

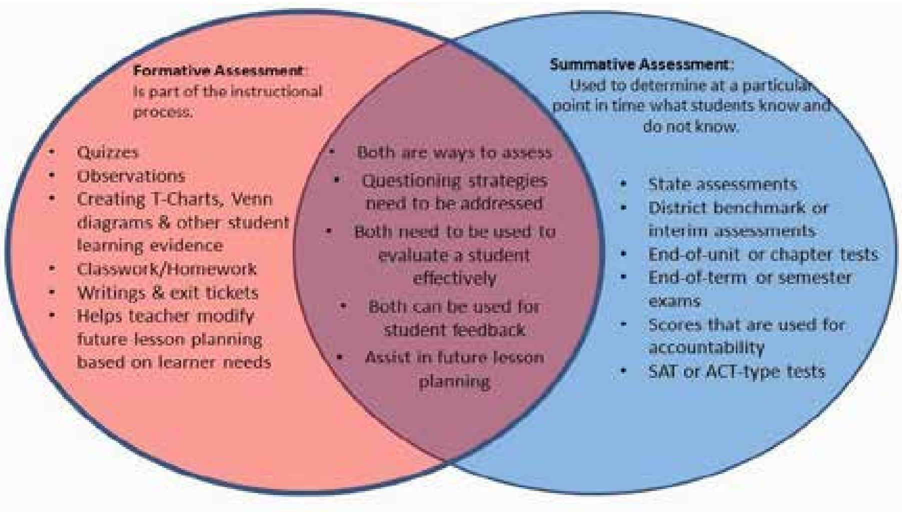


Figure 34: Formative versus Summative Assessment

Source:https://improvingteaching.co.uk/2016/12/11/a-classroom-teachers-guide-to-formative-assessment/

Activity 39

Explain the difference between formative and summative assessment with the cartoon below (Figure 35).

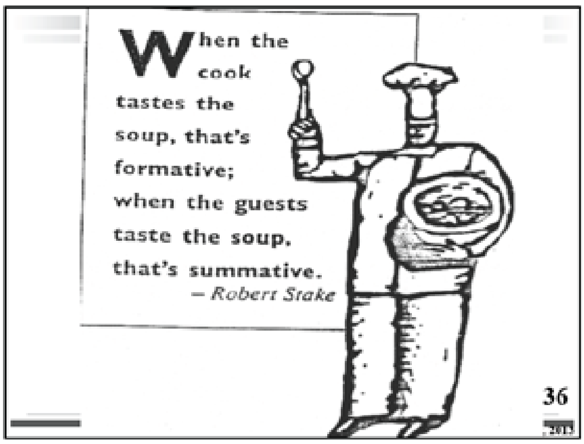


Figure 35: Formative and Summative Assessment

MODULE REFERENCES

Aikman, S., & Underhalter, E. (Eds.). (2007). *Practising Gender Equality in Education*. Retrieved from http://policy-practice.oxfam.org.uk/publications/practising-gender-equality-in-education-115528

Ainscow, M. (2005). Understanding the development of inclusive education system. *Electronic Journal of Research in Educational Psychology*, *3*(7).

Altinyelken, H. K. (2010). Curriculum change in Uganda: Teacher perspectives on the new thematic curriculum. *International Journal of Educational Development*, *30*(2), 151–161.

Askew, M., & Wiliam, D. (1995). *Recent research in mathematics education 5-16* (Vol. 53). London: Ofsted Reviews of Research, HMSO.

Ball, D. L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *The Elementary School Journal*, *90*(4), 449–466.

Ball, D. L. (1997). From the general to the particular: Knowing our own students as learners of mathematics. *The Mathematics Teacher*, *90*(9), 732.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content Knowledge for Teaching What Makes It Special? *Journal of Teacher Education*, *59*(5), 389–407.

Beckmann, S. (2013). *Mathematics for elementary teachers with activity manual*. Pearson.

Beilock, S. L., Gunderson, E. A., Ramirez, G., & Levine, S. C. (2010). Female teachers' math anxiety affects girls' math achievement. *Proceedings of the National Academy of Sciences*, *107*(5), 1860–1863. https://doi.org/10.1073/pnas.0910967107

Benson, S., Addington, S., Arshavsky, N., Al Cuoco, E., Goldenberg, P., & Karnowski, E. (2004). *Ways to Think About Mathematics: Activities and Investigations for Grade 6-12 Teachers* (1st Edition). Corwin Press.

Bentley, K., & Philips, S. (2007). *Teaching Science in CLIL contexts* [Unpublished raw data].

Black, P., & Wiliam, D. (2001). *Inside the black box: Raising standards through classroom assessment* (p. 14). Retrieved from British Educational Research Association website: http://weaeducation.typepad.co.uk/files/blackbox-1.pdf

Black, Paul, & Wiliam, D. (1998). Assessment and classroom learning. *Assessment in Education*, *5*(1), 7–74.

Bloom, B. S. (1968). Learning for Mastery. Instruction and Curriculum. Regional Education Laboratory for the Carolinas and Virginia, Topical Papers and Reprints, Number 1. *Evaluation Comment*, *1*(2), n2.

Bloom, B. S., Hastings, J. T., & Madaus, G. F. (1971). *Handbook on formative and summative assessment of student learning*. New York: McGraw Hill.

Brodie, K. (2014). Learning about learner errors in professional learning communities. *Educational Studies in Mathematics*, *85*(2), 221–239. https://doi.org/10.1007/s10649-013-9507-1

Burns, M. (2015). *About teaching mathematics: A K-8 resource* (4th edition). Houghton Mifflin Harcourt.

Bushart, B. (2016, February 4). Writing Numberless Word Problems. Retrieved from Teaching to the Beat of a Different Drummer website: https://bstockus.wordpress.com/2016/02/04/writing-numberless-word-problems/

Carbonneau, K. J., Marley, S. C., & Selig, J. P. (2013). A meta-analysis of the efficacy of teaching mathematics with concrete manipulatives. *Journal of Educational Psychology*, *105*(2), 380.

Chapin, S. H., O'Connor, C., O'Connor, M. C., & Anderson, N. C. (2009). *Classroom discussions: Using math talk to help students learn, Grades K-6*. Math Solutions.

Chick, H. L., & Harris, K. (2007). Pedagogical content knowledge and the use of examples for teaching ratio. *Proceedings of the 2007 Annual Conference of the Australian Association for Research in Education*, 25–28.

Consuegra, E. (2015). *Gendered teacher-student classroom interactions in secondary education: Perception, reality and professionalism* (Vrije Universiteit Brussel). Retrieved from http://www.rosadoc.be/digidocs/dd-001101_2015_gendered_teacher-student_classroom_interactions_in_secondary_education.pdf

DfES. (2004). *Pedagogy and Practice: Teaching and Learning in Secondary Schools—Unit 10: Group Work* (No. 0433–2004).

Driver, R., Squires, A., Rusworth, P., & Wood-Robinson, V. (2000). *Making sense of secondary science: Research into children's ideas.* New York: Routledge.

Fujii, T., & Stephens, M. (2008). Using number sentences to introduce the idea of variable. *Algebra and Algebraic Thinking in School Mathematics: Seventieth Yearbook*, 127–149.

Hattie, J. (2009). *Visible Learning: A Synthesis of over 800 Meta-Analyses Relating to Achievement*. Retrieved from http://www.learningandteaching.info/teaching/what_works.htm

Hattie, J., & Timperley, H. (2007). The power of feedback. *Review of Educational Research*, *77*(1), 81–112.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking Pedagogical Content Knowledge: Conceptualizing and Measuring Teachers' Topic-Specific Knowledge of Students. *Journal for Research in Mathematics Education*, (4), 372. https://doi.org/10.2307/40539304

Hyde, J. S., Fennema, E., & Lamon, S. J. (1990). *Gender differences in mathematics performance: A meta-analysis.* American Psychological Association.

Kainulainen, M., McMullen, J., & Lehtinen, E. (2017). Early Developmental Trajectories Toward Concepts of Rational Numbers. *Cognition and Instruction*, *35*(1), 4–19. https://doi.org/10.1080/07370008.2016.1251287

Keogh, B., & Naylor, S. (1999). Concept cartoons, teaching and learning in science: An evaluation. *International Journal of Science Education*, *21*, 431–446. https://doi.org/Doi 10.1080/095006999290642

Klibanoff, R. S., Levine, S. C., Huttenlocher, J., Vasilyeva, M., & Hedges, L. V. (2006). Preschool children's mathematical knowledge: The effect of teacher" math talk.". *Developmental Psychology*, *42*(1), 59.

Krathwohl, D. R. (2002). A revision of Bloom's taxonomy: An overview. *Theory into Practice*, *41*(4), 212–218.

Lemov, D. (2015). Teach like a champion 2.0. *San Francisco, CA: Josey-Bass*.

Martino, A. M., & Maher, C. A. (1999). Teacher questioning to promote justification and generalization in mathematics: What research practice has taught us. *The Journal of Mathematical Behavior*, *18*(1), 53–78.

Mason, J. (2008). Making use of children's powers to produce algebraic thinking. *Algebra in the Early Grades*, 57–94.

Mercer, N., & Sams, C. (2006). Teaching Children How to Use Language to Solve Maths Problems. *Language and Education*, *20*(6), 507–528. https://doi.org/10.2167/le678.0

Millgate House Education. (2008). *Concept Cartoons in Mathematics Education* (First Edition). Sandbach, Cheshire, UK: Millgate House Publishers.

MINEDUC. (2015). *Education For All 2015 National Review Report: Rwanda*. Kigali, Rwanda: MINEDUC.

Mlama, P. M. (2005). *Gender Responsive Pedagogy: A Teacher's Handbook*. Presented at the Nairobi, Kenya. Retrieved from http://www.ungei.org/files/FAWE_GRP_ENGLISH_VERSION.pdf

Moss, J., Beatty, R., McNab, S. L., & Eisenband, J. (2006). The potential of geometric sequences to foster young students' ability to generalize in mathematics. *The Annual Meeting of the American Educational Research Association*.

Moss, J., & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and an experimental curriculum. *Journal for Research in Mathematics Education*, 122–147.

National Research Council. (2002). *Helping Children Learn Mathematics*. Retrieved from https://www.nap.edu/catalog/10434/helping-children-learn-mathematics

Nicol, D. (2007). Principles of good assessment and feedback: Theory and practice. *From the REAP International Online Conference on Assessment Design for Learner Responsibility, 29-31 May 2007*. Presented at the REAP International Online Conference on Assessment Design for Learner Responsibility, 29-31 May 2007. Retrieved from http://www.reap.ac.uk/reap/public/papers//Principles_of_good_assessment_and_feedback.pdf

NISR. (2015). *Integrated Household Living Conditions Survey (EICV)* (No. 4). Retrieved from National Institute of Statistics in Rwanda website: http://www.statistics.gov.rw/survey/integrated-household-living-conditions-survey-eicv

Nsengimana, T., Habimana, S., & Mutarutinya, V. (2017). Mathematics and science teachers' understanding and practices of learner-centred education in nine secondary schools from three districts in Rwanda. *Rwandan Journal of Education*, *4*(1), 55–68.

Nsengimana, T., Ozawa, H., & Chikamori, K. (2014). The implementation of the new lower secondary science curriculum in three schools in Rwanda. *African Journal of Research in Mathematics, Science and Technology Education*, *18*(1), 75–86.

Nunes, T., & Bryant, P. (1996). *Children doing mathematics*. Wiley-Blackwell.

OECD. (2006). *Assessing Scientific, Reading and Mathematical Literacy: A Framework for PISA 2006* (p. 187). Retrieved from OECD website: http://www.oecd.org/edu/school/assessingscientificreadingandmathematicalliteracyaframeworkforpisa2006.htm

OECD. (2015). *The ABC of Gender Equality in Education*. Retrieved from http://www.oecd-ilibrary.org/content/book/9789264229945-en

Page, A. (1994). Helping students understand subtraction. *Teaching Children Mathematics*, *1*(3), 140–144.

Polya, G. (1945). *How to solve it: A new aspect of mathematical method*. Princeton university press.

REB. (2015). *Competence-Based Curriculum Pre-Primary to Upper Secondary*. Retrieved from MINEDUC website: http://reb.rw/fileadmin/competence_based_curriculum/syllabi/CURRICULUM_FRAMEWORK__FINAL_VERSION_PRINTED.compressed.pdf

REB. (2017). *Literacy, Language and Learning Initiative (L3): National Fluency And Mathematics Assessment Of Rwandan Schools: Endline Report*. Retrieved from USAID, EDC website: http://l3.edc.org/documents/EDC-L3-Endline-Evaluation.pdf

Richardson, K. (2012). *How Children Learn Number Concepts: A Guide to the Critical Learning Phases*. Bellingham, Wash.: Math Perspectives Teacher Development Center.

Rwanda Education Board. (2015). *Summary of Curriculum Framework Pre-Primary to Upper Secondary*. Retrieved from Rwanda Education Board website: http://reb.rw/fileadmin/competence_based_curriculum/syllabi/CURRICULUM_FRAMEWORK_FINAL_PRINTED.compressed.pdf

Save the Children, Mureke Dusome project. (2017). *Parent-School Partnerships for Education Toolkit: Training Modules for Sector Education Officers*. Save The Children, REB, USAID.

Schweisfurth, M. (2011). Learner-centred education in developing country contexts: From solution to problem? *International Journal of Educational Development*, *31*(5), 419–426.

Shulman, L. S. (1986). Those Who Understand: A Conception of Teacher Knowledge. *American Educator*, *10*(1).

Smith III, J. P., Disessa, A. A., & Roschelle, J. (1994). Misconceptions reconceived: A constructivist analysis of knowledge in transition. *The Journal of the Learning Sciences*, *3*(2), 115–163.

Stewart, G. (2014). *Promoting & Managing Effective Collaborative Group Work*. Retrieved from Belfast Education and Library Board.

Subrahmanian, R. (2005). Gender equality in education: Definitions and measurements. *International Journal of Educational Development*, *25*(4), 395–407.

Swan, M. (2005). *Standards Unit-Improving learning in mathematics: Challenges and strategies*.

Sweller, J., & Cooper, G. A. (1985). The Use of Worked Examples as a Substitute for Problem Solving in Learning Algebra. *Cognition and Instruction*, *2*(1), 59–89. https://doi.org/10.1207/s1532690xci0201_3

The National Council of Teachers of Mathematics (NCTM). (2007). *What Do We Know about the Teaching and Learning of Algebra in the Elementary Grades?*

UNICEF. (2017). *UNICEF Gender Action Plan, 2018–2021*. Retrieved from UNICEF website: https://www.unicef.org/gender/files/2018-2021-Gender_Action_Plan-Rev.1.pdf

Vamvakoussi, X., Van Dooren, W., & Verschaffel, L. (2012). Naturally biased? In search for reaction time evidence for a natural number bias in adults. *The Journal of Mathematical Behavior*, *31*(3), 344–355.

Van de Walle, J. A., Karp, K. S., & Williams, J. M. B. (2015). *Elementary and middle school mathematics. Teaching development* (9th Edition). Boston: Pearson.

Vavrus, F., Thomas, M., & Bartlett, L. (2011). *Ensuring quality by attending to inquiry: Learner-centered pedagogy in sub-Saharan Africa*. UNESCO-IICBA Addis Ababa, Éthiopie.

Weimer, M. (2012). Five characteristics of learner-centered teaching. *Faculty Focus*.

Wiliam, D. (2016, September 2). The 9 things every teacher should know. Retrieved 8 March 2017, from TES website: https://www.tes.com/us/news/breaking-views/9-things-every-teacher-should-know

Zuze, T. L., & Lee, V. E. (2007). *Gender Equity in Mathematics Achievement in East African Primary Schools: Context Counts* (No. 15). Cape Town, South Africa: Southern Africa Labour and Development Research Unit.

mastercard foundation Leaders in Teaching | Belgium partner in development | Flanders State of the Art