Continuous Professional Development Certificate in Educational Mentorship and Coaching for Mathematics Teachers (CPD-CEMCMT)

APPENDIX

Module 2

3rd Edition

Pedagogical Content Knowledge and Gender in Mathematics Education

February 2020

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MODULE TWO

APPENDIX

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Section 1: Numbers and Operations

Number Lines

Number lines are a useful tool to help learners develop a sense of the meaning of numbers in the early primary years. They are also useful to gradually let learners develop the concept of place value, one of the most important concepts in primary maths. A number line is a graduated straight line that serves as representation for real numbers. In this section, we introduce a variety of questions and activities with number lines.

1. Comparing numbers on a number line

Number lines are useful to develop a sense of the relative size of numbers with learners.

Which number is bigger?

2 or 5 11 or 9 -2 or 5 -5 or 2

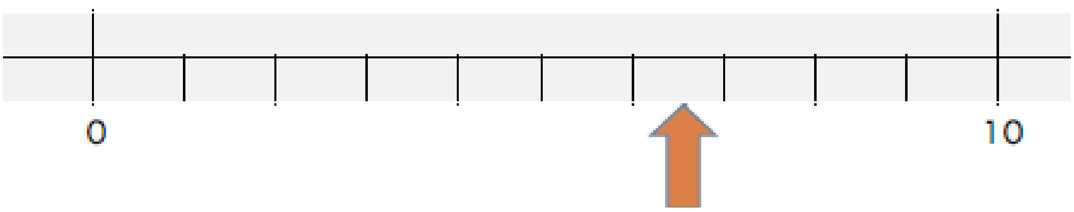
How do we decide? Place both numbers on a number line:

4.37 or 3.5737 1.8 or 1.08 -4.3 or 3.7

2. Using an empty number line

How do we label the number the arrow is pointing at? How do we use place value to help us with the label?

Label 3,2 5,9 6,7 on this number line. Label a new number line with 6.7 6.17 and 6.71.



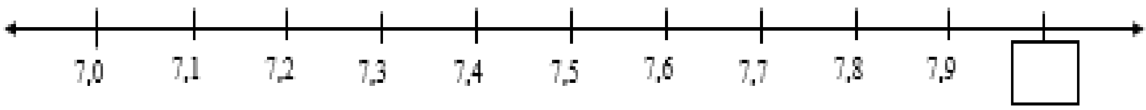
3. Use an empty number line to show that:

Number lines can be used to develop learners' understanding in place value, such as the meaning of unit, tenths, hundredths and thousandths.

- 1.7 lies between which two successive units?
- 1.73 lies between which two successive tenths?
- 1.738 lies between which two successive hundredths?

4. Dealing with misunderstandings in number sense

This activity exposes a frequent misconception with learners: that 7.10 is bigger than 7.9. Edouard wrote 7.10 in the empty box on the number line below. Why would he write this? Describe how you could help Edouard to find the correct answer.



5. Subtraction and use of number lines

Number lines can be used to teach mental calculation strategies for addition and subtraction.

Jeanne and Thomas want to calculate 253 – 99 by first calculating

253 – 100 = 153

Jeanne says that they must now subtract 1 from 153, but Thomas says that they must add 1 to 153.

Draw a number line to help you explain who is right and why.

Place Value

Place value is a key concept in primary mathematics, as it forms the basis for numbers and operations. The following activities can help learners acquire a sound understanding of place value.

6. Counting Strategies

The first step in developing number sense is developing strategies for counting objects faster. In this activity, you need toothpicks (or another counting object) and a die. The activity helps learners explore how they can count faster by grouping. It is a good way to introduce the concept of tens. Counting collections also introduces ideas about how the place value system helps counting.

In small groups:

1. Toss the die, then multiply the number by 6
2. Represent this total with bundled toothpicks
3. Toss the die again, then multiply the number by 5
4. Represent this total with bundled toothpicks

Combine the two bundles (or dried beans, bottle caps…) and calculate the overall total. After students have done their counting, discuss strategies children used for counting. Was it easier to count by 2s? By 10s? What other strategies did learners use? Did all the groups who counted the same thing get the same answer? Which counting methods are most accurate? Which are easiest?

7. Rounding off to the nearest… activity

Many learners think that rounding off means always rounding to a higher number. Using a number line for the exercise below can help learners understand that rounding off can result in a lower of in a higher number.

Round 34.617 to:

- The nearest five
- The nearest ten
- The nearest hundred
- The nearest tenth
- The nearest hundredth

Round 12.56 to these values as well.

Round 999 253. 34 to all these values as well.

8. Draw a secret number

Draw a number line with a 0 and 200 at opposite ends of your line. Mark a point with a question mark that corresponds with your secret number. Estimate the position the best you can. Students guess your secret number. For each guess, place and label a mark on the line that corresponds with the number guessed.

Continue marking each guess until your secret number is discovered. You can vary in the endpoints. For example, try 0 and 1000, 200 and 300 or 500 and 1000. It is important that you mark the guesses of the learners. Labelling those numbers at the correct locations will support students' reasoning in the process of identifying the secret number.

After you played the game with the whole class, learners can play it in small groups.

9. Close, far and in between

This activity is useful for developing learners' sense of place value and their skills in basic operations (Van de Walle et al., 2015).

Put any three numbers on the board. Use numbers that are appropriate to the learners' level (for example, 257, 344 and 405). Starting from these 3 numbers, ask questions such as the following and encourage discussion, for example through voting.

- Which two are closest? Why?
- Which is closest to (200)? To (450)?
- Name a number between (257) and (344).
- Name a multiple of 25 between (257) and (344).
- Name a number that is more than all these numbers.
- About how far apart are 257 and 500? 257 and 5000?

10. Which one doesn't belong?

Activities like this one are useful to organize mathematical conversations. Let learners with different answers reason about their answer. You can also let learners work in groups to develop their own "which one doesn't belong" questions and let them solve each other's questions.

Provide students with lists of numbers and asking them to argue why one of the numbers doesn't belong to the list. There can be different valid solutions, as long as the arguments are sound. For example,

1/2	5/3	2/10	1/5

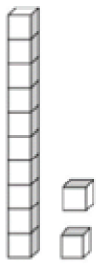
Another example:

0.25	¾	0.8	0.5

Another example:

$$\frac{11}{10} \qquad 1.10$$

10 tenths 9 hundredths 10 thousandths



Addition and Subtraction

11. Drawing number lines for addition and subtraction problems

It is important to help learners notice the different situations in which to use subtraction and the language that you use when talking about subtraction (Page, 1994). Many students in early grades only know the take-away meaning for subtraction. For problems such as 100 – 3 = ___. Thinking in terms of take-away serves many students well. A popular strategy is to start from 100 and count down (99, 98, 97), often using fingers. For problems like 100 – 3, this way of reasoning is good, because the subtrahend is small (the student must take away only 3). However, in 201 – 199, the difference is small, but the subtrahend is large. In these situations, thinking about subtraction as take-away can be highly inefficient, whereas thinking of the difference as the distance between the given numbers is more useful.

Drawing number lines help learners view addition and **subtraction problems as distances between numbers** and make connections between the ideas of addition and subtraction, counting forward and backward and even linear measurement. It reinforces their insights in the relationships between numbers and their mental mathematics competences. Seeing differences as distances between numbers also works better when working with negative integers, for example for 3 – (–5). Finally, reasoning about differences in terms of distance is good preparation for the transition from arithmetic to algebra.

Use a number line posted on the wall of your classroom when discussing subtraction problems and strategies or make it a habit to draw a number line with addition and subtraction problems.

For example, consider 81 -29 (Figure 1).

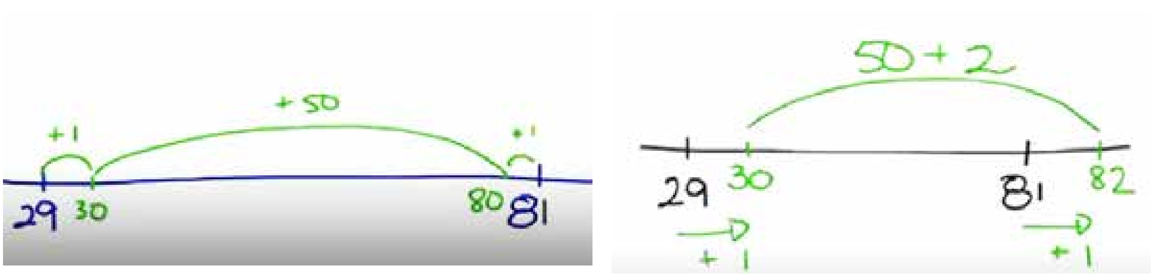


Figure 1: Using a number line for subtractions

12. Use a variety of word problem types for addition and subtraction

It is important that learners can recognize various problem types (Carpenter & Lehrer, 1999) in word problems, including

- Joining Situations (variations: result unknown, change unknown, start unknown)
- Separating Situations (variations: result unknown, change unknown, start unknown)
- Part-Part-Whole Situations (Whole Unknown, Part Unknown)
- Comparison Situations (Difference Unknown, Larger Quantity Unknown, Smaller Quantity Unknown)

For example, the following word problems contain a combination of addition and subtraction situations. Discuss with your learners <u>which are addition and subtraction problems</u> and why. Use number lines (and let them draw number lines) to visualize each problem.

1. Eline has 23 apples. She got 18 more apples. How many does she have now?
2. Pierre has 23 apples and 29 bananas. How many pieces of fruit does he have?
3. Alex had 38 apples. He gave away 19 apples. How many does he have now?
4. Chris has 17 tomatoes. Pierre has 15 tomatoes more than Chris. How many tomatoes does Pierre have?
5. Fabrice has 16 apples. Benny has 46 apples. How many fewer apples does Fabrice have than Benny?
6. Elsie has 12 mangoes. How many more mangoes does she need to have 30 mangoes altogether?
7. Marie has 12 red triangles and 3 blue triangles. How many more red triangles does Marie have than blue triangles?
8. Farida had some pencils. After she got 5 more pencils, Farida had 22 pencils altogether. How many pencils did Farida get?
9. Eugene is reading a book that has 462 pages. He has 148 pages left to read. How many pages has he read?
10. In a bag of 74 marbles, 45 belong to Pierre and the other belong to Marie. How many marbles does Marie have?

13. Let students create their own word problems

A powerful, inclusive exercise is to let students create their own word problem based on a given addition or subtraction. Discuss the variety of responses with the learners and try to include different types of addition and subtraction problems in the discussion. Apart from developing learners' problem-solving skills, this kind of exercises also strengthens their correct use of mathematical language. You can extend this exercise to include multiplication and division, as well as decimal and negative numbers.

1. Create a problem for 5+ 3, 12 -4…
2. Create change problems and part-part-whole problems for 5+ 3, 12 -4
3. Create a compare problem (comparison of a larger quantity and a smaller quantity) for 5+ 3, 12 -4

Box: Further Reading

https://buildingmathematicians.wordpress.com/2016/11/25/subtracting-integers-do-you-see-it-as-removal-or-difference/)

http://mathforlove.com/lesson/pyramid-puzzles/

Multiplication and Division

Introduction

The key difference between additive and multiplicative reasoning is that additive reasoning is based on thinking about how quantities are related in terms of how much more or less, whereas multiplicative reasoning is based on thinking about how quantities are related in terms of **how many times more or less** (Beckmann, 2013).

In the early grades, the emphasis should be on making sense of multiplication and division situations and represent them. Make explicit connections between skip counting (addition) and multiplication situations. Use various multiplicative situations like scaling up (e.g. doubling or 'three times as many children') and scaling down (halving or 'a quarter of the chocolate bar') and linking them to students' daily lives.

In upper primary, the emphasis should be on introducing various models that support children with multiplication and division. In this, it is important to focus on sense-making (conceptual understanding), rather than only on the procedures. Students need to get familiar with various situations that can be modelled through multiplication (as repeated addition, rate, scaling) and division (sharing and grouping) (See Table 1).

Table 1: Meanings of Multiplication

SIMPLE RATIOS	If a mango costs 500 Frw, how much will I pay for 5 mangoes? This question has an implicit 'per item' built into it: 500 Frw per mango, and so is a very simple proportion problem: 1 (mango) is to 500 (Frw) as 5 (mangoes) is to 2500 (Frw).
REPEATED ADDITION	On Monday Michel saved 800 Frw. On Tuesday, he saved 800 Frw and on Wednesday he saved 800 Frw. How much did Michel save altogether?
CARTESIAN PRODUCT	Tom has 4 t-shirts and 3 pairs of jeans. How many days can he go out and wear a different combination of t-shirt and jeans?
SCALING MEASUREMENTS	On Monday, Alice's beanstalk was 15 cm tall. On Friday, it was 5 times as tall. How tall was the beanstalk on Friday? How many times bigger (or smaller)?
MULTIPLE PROPORTIONS	A jug of milk provides enough milk to fill five saucers. A pail of milk will fill four jugs. How many saucers of milk can be filled from a pail of milk?

14. Using various visual models of multiplication situations

Introduce various visual interpretations of multiplication. Depending on the problem, one representation might be more suitable than others. Multiplication situations can be represented by an area model, a double number line or a simple number line.

1. Area Model

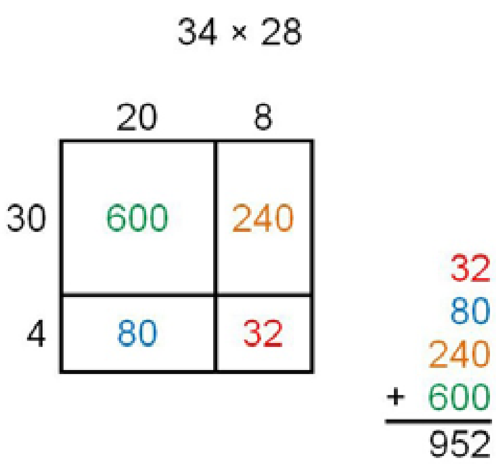


Figure 2: Area model for multiplication

2. Double number line

For example:

1. Petrol costs 1000 Frw for 1 litre. What is the cost of 15 litres?

2. If James earned 12 000 Frw in 8 hours, how much would he earn in 3 hours?

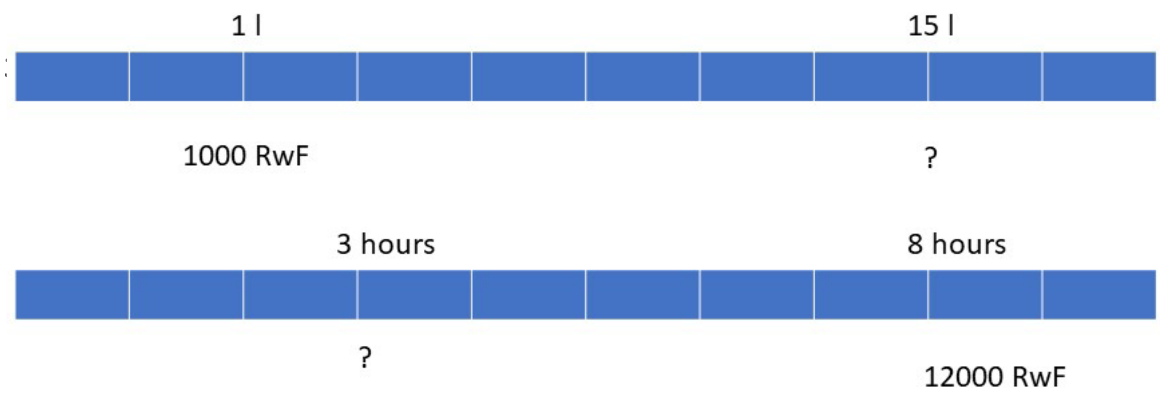


Figure 3: Using double number lines to represent multiplications

4. Scaling on a number line

Last month, Fabien had 14 marbles. Now he has 3 times as many marbles. How many marbles does he have?

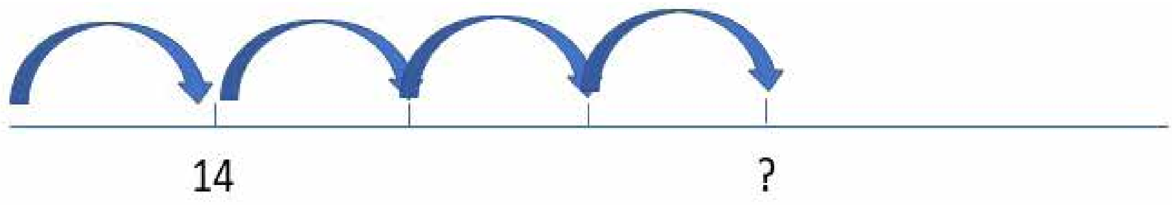


Figure 4: Scaling on a number line

Clarifying the relation between multiplication and division

For their conceptual understanding, it is important to let learners discover the relation between multiplication and division. For example, 6 bags each hold 7 mangoes. How many mangoes are there altogether? This is an example of a multiplication question. 42 mangoes are shared equally into 6 bags. How many mangoes does each bag contain? Now, the problem has become a division problem (see Table 2**Error! Reference source not found.**).

Table 2: Relation between multiplication and division

Bags	Mangoes
1	7
6	?

Multiplication

Bags	Mangoes
1	?
6	42

Division as sharing

Let learners practise reformulating multiplication problems so they become a division problem. Let them distinguish between division as sharing (number of groups is known and size of each group is unknown) and grouping (number of groups is unknown, but size of each group is known). For example, write a simple word problem and make a math drawing to help children understand what 10 ÷ 2 means (Beckmann, 2013).

Many students find it difficult to understand the "how many groups" interpretation of division. However, this is the model that makes the most sense for the division of fractions (Beckmann, 2013). This interpretation of division can also impact a student's ability to be successful with long division. For example, students need to be able to think, "How many groups of 30 are there in 1429?"

15. Identifying patterns in multiplications

Although learners should be able to solve multiplication problems with the standard algorithm, it is useful to let them also look at multiplications (and divisions) without immediately using this algorithm. Sometimes, there are easier and faster ways to solve a multiplication problem. It is good when learners master different procedures to solve a problem. Not only can they select a procedure according to the context, they can also verify a result obtained with one procedure by using another procedure.

Present learners with these **sequences of multiplications**. How can learners solve each one based on the result of the previous one? It is important that you give these problems in series, so learners can discover the relations between them. Discuss the relations with your learners. Let learners <u>use correct mathematical language to describe patterns and relationships</u> they notice.

48 x 26	448 x 25	448 x 2,5	2,5 x 8,4
10 x 8	2 x 8	2 x 8	6 x 16
6 x 10	6 x 40	6 x 39	6 x 41
18 x 10	18 x 9	19 x 18	21 x 18
6 x 8	12 x 4	3 x 16	32 x 1,5
21 ÷ 7	70 ÷ 7	91 ÷ 7	42 ÷ 14
24 ÷ 2	24 ÷ 4	24 ÷ 8	24 ÷ 16
12 ÷ 2	24 ÷ 4	48 ÷ 8	48 ÷ 16
160 ÷ 16	320 ÷ 16	640 ÷ 16	1240 ÷16
6 x 20	6 x 100	6 x 120	6 x 119

Other examples to practise reasoning skills in mathematics are estimation questions:

- What is the rough cost (no detailed calculation) of 21 cans of coke costing 360 Frw each?

Other examples:

- 2.6 x 4,8
- 1.26 x 0.5
- Estimate how much is 102 x 102.
- Estimate how much is 102 x 98.
- Estimate how much is 21/23 x 8/9

16. Numbers and operations game

This activity can be used as a game to practise learners' skills in basic operations. You can make the sequences as difficult as you like.

Given a set of 5 numbers, try to get as close as possible to the number on the top by using addition, subtraction and multiplication with the numbers below:

30	**45**	**61**
9	9	9
1	6	8
3	11	7
7	3	3
4	2	11

17. Word problems with multiplication and division

Here are some examples of simple word problems for multiplication and division. It is good to mix word problems (different meanings of multiplication and division), so that learners are stimulated to think for each problem. Stimulate learners to make drawings of the word problem. Let them explain the problems to each other and let them construct their own problems.

- 6 bags each hold 14 mangoes. How many mangoes are there altogether?
- 1/3 of the children in a class have a white shirt. ½ of those children also have black trousers. How many children in the class have black trousers and a white shirt?
- A recipe needs 2/3 kg of sugar. You only want to make ½ of the recipe. How much sugar should you use?
- A pharmacist has 7,5 l of a cough mixture. She wants to distribute it in bottles of 0,25 l each. How many bottles can she fill with the mixture?
- You have 2/3 of a pie left over from Christmas. You want to give 1/2 of it to your sister. How much of the whole pie will this be?
- A pharmacist has 2.5 l of a cough mixture that she wants to distribute equally over 6 bottles. How much can she put in each bottle?
- Elisa prepared 12.4 l of mango juice. She wants to distribute the juice equally among 30 children. How much juice will each child get?
- A kilogram of potatoes costs 400 Francs. How much will you pay if you buy 6 kg of potatoes?
- A water tank holds 235 l of water. Albert wants to divide the water into pots of 5 l. How many pots can he fill?
- The district has a fence of 740 m. It wants to plant a tree every 12 m. How many trees can the district plant?
- The government donates 52 198 books to 5 schools with 9 classes each. It wants to give each class the same number of books. How many books will each class receive?
- The village has harvested 14 820 kg of beans. It wants to give families 250 kg of beans. How many families can be given beans?

- A field is 320 metres long and 78 metres wide. What is the field's perimeter? What is its area?
- 6 lengths of fencing are each 8 metres long. How much fencing is there altogether?
- A litre of petrol costs 940 Frw. How much would 8 litres cost?
- A man's shadow is 3.5 times as long as his height. If he is 1.73 metres tall, how long is his shadow?
- How many teams of 15 can be formed from 263 children?
- How many coaches (allowed to carry a maximum of 42 passengers) will be needed to transport all the children?

18. Using place value and number lines in calculations

This activity develops learners' skills in using place value for mental mathematics. These basic operations can be solved with the standard algorithms but can also be solved more quickly using place value and number lines. When students are familiar with the strategy, they can use the number lines only in their head instead of drawing them. It is important that learners are familiar with different strategies to solve basic operations problems. In some cases (such as with the examples below), using place value and flexible grouping strategies involving the use of 5/10 ("friendly numbers") is quicker than using the standard algorithm. Encourage learners to **notice when this strategy is helpful**, depending on the numbers in the problem.

- 199 + 199
- 265 + 197
- 199 + 299
- 104+98
- 4265 + 147 + 949
- 4307 – 609
- 48 x 6
- 98 x 19
- 642 ÷ 3

Section 2: Fractions, Decimal Numbers and Percentage

Introduction

When first learning to count, children often use their hands or physical objects as tools to help the process of counting with whole numbers. However, when the time comes to expand one's number concept, the safety of the fingers or physical objects reaches an end. Linked to this, there is the phenomenon of **Natural Number Bias** in which students continue applying the rules of natural numbers (e.g., larger digits mean larger numbers) to rational numbers (e.g., larger digits can also be an indicator of smaller numbers: 2/3 > 5/9; 2.20>2.025), even when these rules conflict with each other (Vamvakoussi et al., 2012). Research shows that Natural Number Bias is the biggest difficulty to overcome in understanding rational number concepts (Kainulainen et al., 2017)

Fractions, decimal numbers and percentages require big changes in learners' concept of numbers in aspects as their symbolic representation (discrete numbers vs. fractions, decimals, percentages), their size (larger quantity of digits makes number larger vs. larger quantity of digits can make number smaller or larger), and operations such as multiplication (makes numbers larger vs. can make numbers larger or smaller) and division (makes numbers smaller vs. can make numbers smaller or larger).

For many teachers, the topic of rational numbers is a difficult topic for teaching, because children's ways of understanding rational numbers may be very difficult for teachers to observe (Moss & Case, 1999; Nunes & Bryant, 1996). Students may go through school without understanding the qualities of fractions, without anyone noticing it. Nevertheless, fractions are a <u>key concept in primary mathematics</u>. They form the basis for understanding decimals and percentages, algebra and probability.

Ideas to introduce fractions in the early grades

Introduce different situations where learners need to share something equally. Use different units (chocolate bars, bananas, pencils) and contexts.

For example,

- Elsie and David want to share 3 chocolate bars equally. Show them how to do it.
- Elsie, David and Laurence want to share 4 chocolate bars equally. Show them how to do it.
- Elsie, David, Laurence and Fabien want to share 5 chocolate bars equally. Show them how to do it.

Comments:

- There are more objects (chocolate bars) than children
- Allow children to make sense of the situation and to draw the solution – they do not need the fraction names or notations yet.
- Discuss the different plans that children in the group made.
- Reason for the choice of chocolate bars: rectangular objects.
- Young children have already been introduced to the idea of a 'fraction' before formally learning about the concept in school using language such as: 'a small piece', 'a little bit'.
- Fractions should be introduced to young children using real problems that involve dividing or breaking – which support them to come up with their own solutions. Fractional terms like one half can be introduced as the need arises.

Introduce various possibilities for what a unit is:

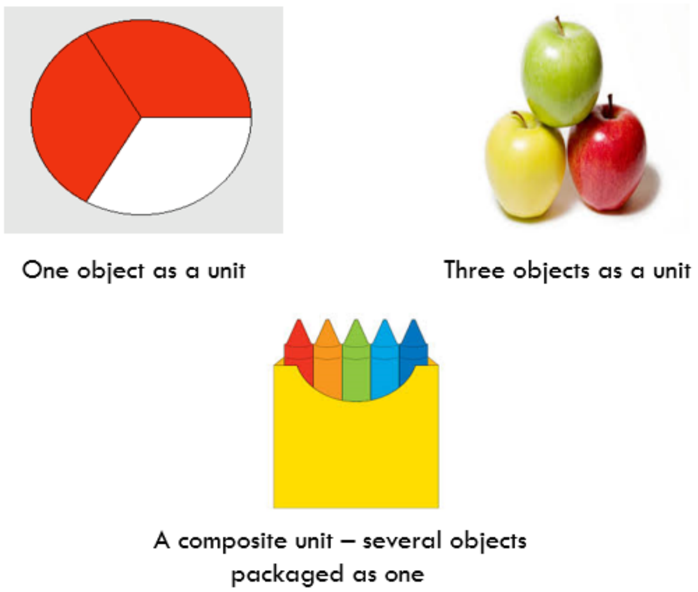


Figure 5: Various units for fractions

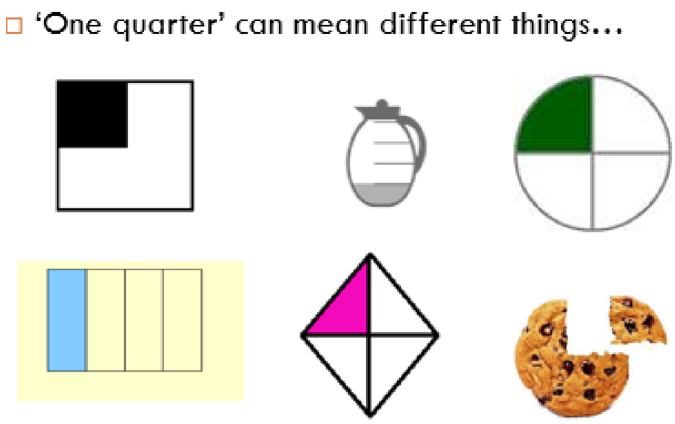


Figure 6: Various meanings of one quarter

Why do learners find fractions difficult?

1. They express a relative rather than fixed amount
2. The same fraction can refer to different quantities
3. The same quantity can be expressed by different equivalent fractions
4. Any fraction can refer to objects, quantities or shapes

5. The rules for whole numbers do not always apply
6. A fraction can be a part of a shape or shapes, a part of a set of discrete objects or a position on a number line (a number in its own right).

¾ can mean many things:

1. Three parts of a pizza cut into four equal parts.
2. The result of four hungry children equally sharing three pizzas.
3. The fraction of counters that are red if there are four counters on the table, three red and one white.
4. The likelihood of turning over an even number card if cards with 1, 2, 4 and 6 on are face down on the table.
5. The fraction of a puppy's length if it is 12 cm long and its mother is 16 cm long.

¾ can also be expressed in (an infinite) number of different ways (equivalent fractions), including,

- 6/8
- 30/40
- 0.75
- 75%.

Use of Double Number Bars and Ratio Tables

Double number bars can be used when two quantities that are in relation to each other are measured in different units.

For example, I buy 2 mangoes for 900 Frw. How many mangoes can I buy with 1800 Frw?

kg	2	10	1		5	
RwF	900			1800		

Double number bars help children make the move from additive to multiplicative reasoning.

They can also be used to link multiplication and division. For example:

I am putting apples into bags. There are six apples in each bag. I fill seven bags. How many apples is that? (Multiplication).

I am putting apples into bags. There are six apples in each bag. I have 42 apples. How many bags can I fill? (Division as repeated subtraction/grouping)

Bags	Apples
1	6
7	?

Bags	Apples
1	6
?	42

Rene was putting photos into an album. He put the same number on each page. He put 6 photos on each page. He had 42 photos. How many pages did he fill?

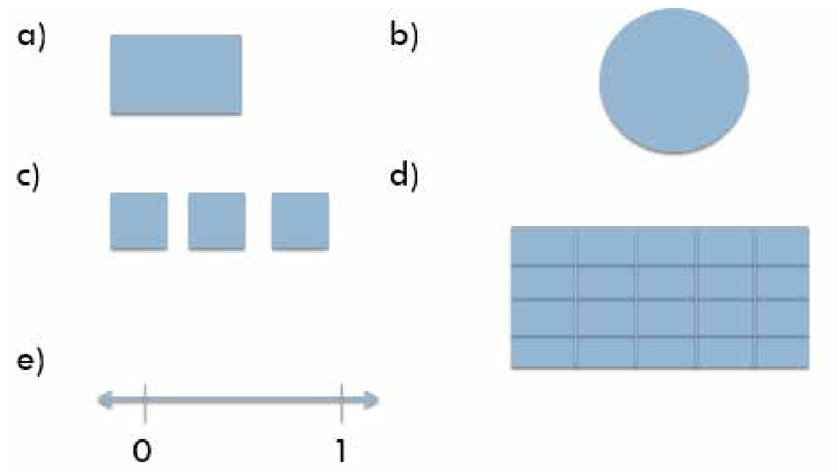
Sarah was putting stickers into an album. She put the same number on each page. She filled 7 pages. She had 35 stickers. How many stickers did she put in each page?

Pages	Photos
1	6
?	42

Pages	Stickers
1	?
5	35

Activities

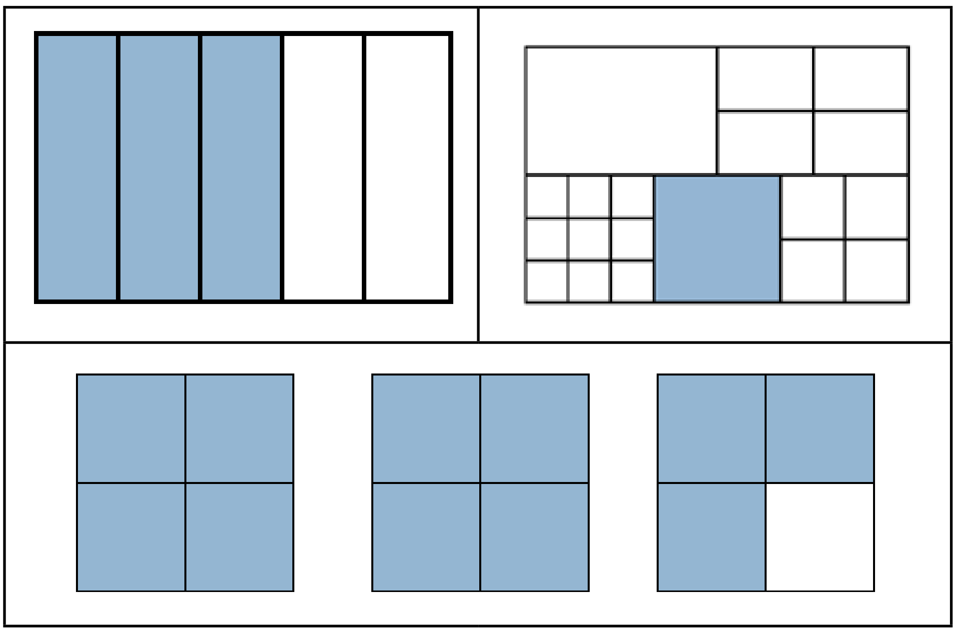
1. Sketch these diagrams and shade in one tenth of the diagram in each case. Sketch the diagrams again and shade one fifth of the diagram. How can we write the answer in each case?

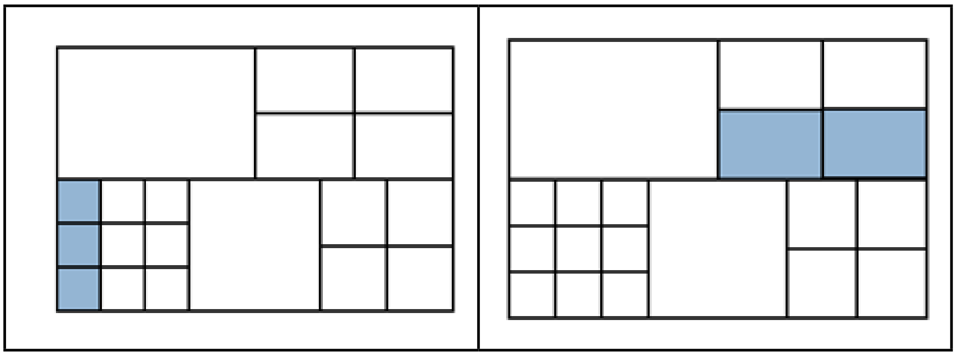


This exercise introduces various units of fractions. In discussing the cases, you can move between part/whole relation, fraction, decimal notation and %.

2. What fraction is coloured blue?

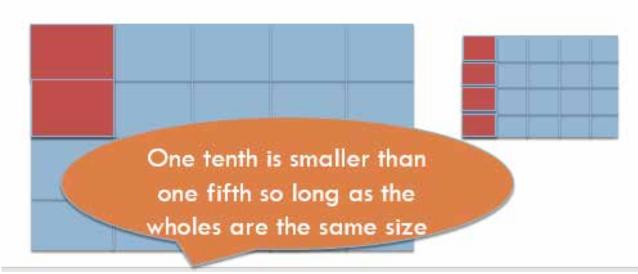
Use representations such as the ones below to familiarize students with various fraction units.





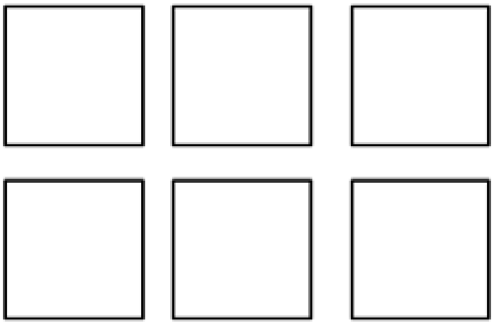
3. One tenth is always smaller than one fifth. Correct?

This kind of question lets students actively engage with and discuss frequent misconceptions, based on differences between fractions and integers (Beckmann, 2013).



4. Shade a quarter

Use various shapes and units and let learners shade various fractions.



5. Ordering fractions

Letting learners order fractions from smallest to largest is a good exercise to develop their understanding of fractions. You can do this as a think-pair-share. During the class discussion, stimulate reasoning by students. Examples are:

1/2, 1/5, 1/3, 1/7 and 1/10

1/4, 11/6, 3/8, 1/16 and 3/4

3/4, 5/3, 6/7 and 1/6

6. Fractions and Proportions

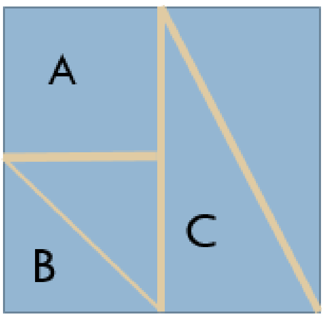
What fraction of the square do A, B and C represent? What fraction do we get when we put A and C together? A and B together? B and C together?

How many times bigger is A than B?

A = __ of B

B = __ of A

What fraction is half of B? And half again? What fraction is one third of A? What proportion of the whole do A and B make together?



This question directly aims to address common misconceptions about fractions – that equal shares means identical appearance, or same shape, rather than same proportion of the overall unit. In this question, the areas A and C can be represented by the same fraction (1/4).

7. Using a Fraction Wall to relate fractions to decimal numbers and percentages

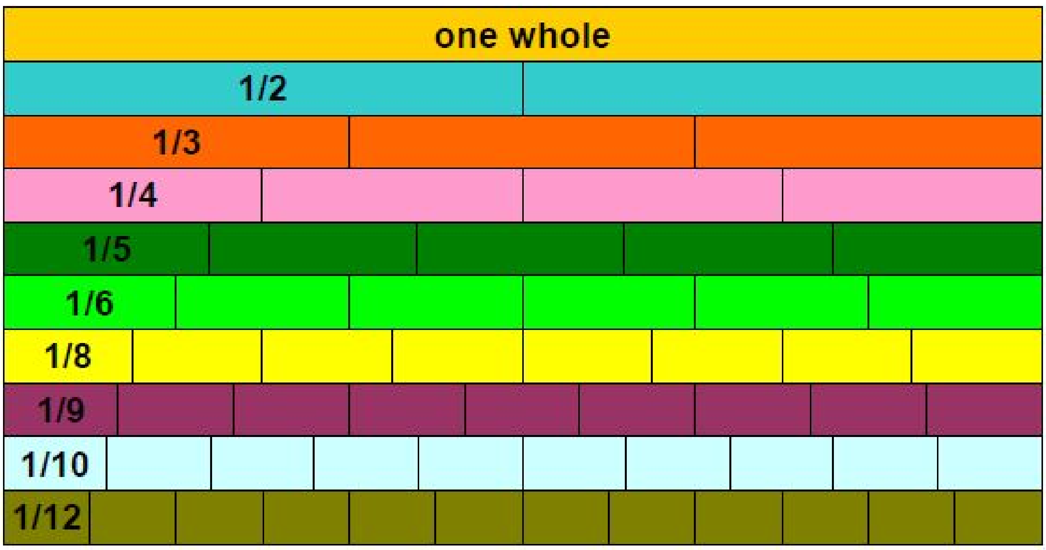
Fractions walls are a useful instrument to practise with students the operations with fractions, and the relations between fractions, decimal numbers and percentages.

One possibility is to let students write each fraction as a decimal number and a percentage. Discuss which ones are easy to write as decimals and %. Why are they easy? Secondly, you can pose questions like: Can you find a fraction/decimal in between two other fractions and decimals? You can let learners play a game. One learner chooses two fractions or decimals on your line. The other learner must name a fraction between the two. For example, find a fraction between 2/5 and 3/5, between 5/8 and 6/8, between 1/3 and ¼.

Using a fraction wall, let learners solve problems like:

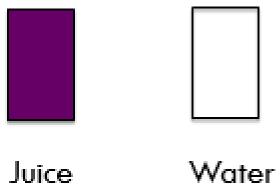
Which fraction is bigger?

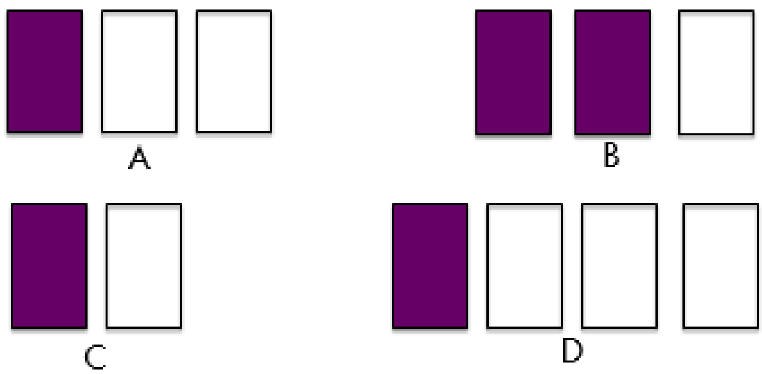
- 5/6 or 4/6
- 3/7 or 3/8
- 7/8 or 8/9



8. Thinking in proportions: making juice example

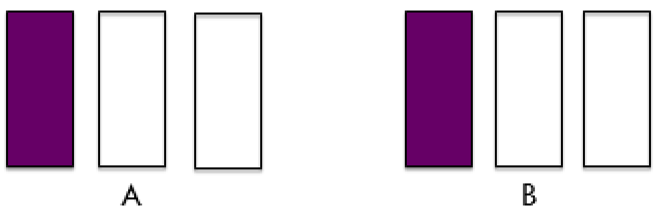
There are four mixtures of juice and water (A, B, C, D). Which juice is the tastiest (which mixture has proportionally the most juice in it)?

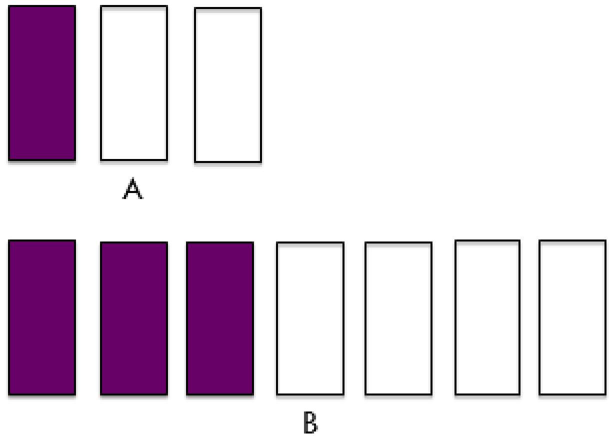




With this question teachers can link part-part language of ratios to part-whole language of fractions. You can take the question further. What happens when we make more juice (see figure below)? Do both juices still taste the same?

A and B make some juice. 1-part juice to 2 parts water. But B decides he wants some more juice, so he adds two more parts of juice and 2 more parts of water to his juice. Will his juice still taste the same as the first juice he made?





9. Word problem: Which is better?

Afrodis got 17/20 on a mathematics test. He got 22/25 on a science test. Joe says he is as good at mathematics as he is at science because he got 3 questions wrong on each test. Draw a diagram to show whether Afrodis is correct.

10. Word Problems on proportions

A muffin recipe needs flour and milk in the ratio 9: 2. How many cups of milk would be needed to go with 21 cups of flour?

Therese has 8 tins of cool drink. How many glasses can she fill from the 8 tins, if one glass takes exactly three fifths of a tin? Use a diagram to work out the answer.

11. Proportions involving Fractions

Use double number bars to work out your answers for the missing values in the table

servings	4	2	8	6	10	16
Flour (cups)	¾					
Butter (cups)	¼					
Icing sugar (tbs)	3					
Water (tsp)	2½					
Yoghurt (cups)	1 and 1/3					

12. Fill in the blanks

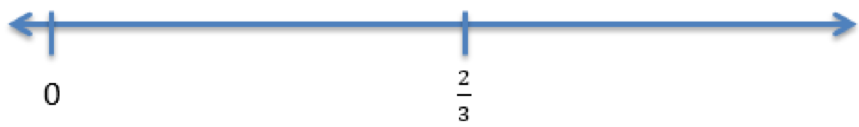
The questions below are examples of open questions that learners can solve at different levels. Therefore, they enable differentiation at the task level.

____ is double ____ ______ is half of ______ _____ is five times _____ _____ is one fifth of ______ Make up another linked pair	____ is ¼ of ____ 8 is ______ (fraction) of ______ _____ is 3/5 of _____ ______ is ______ (fraction) of 9 Make up two more like this.
____ is 25% of ____ 8 is ______ % of ______ _____ is 125% of _____ _____ is _____ % of 340 Make up two more like this.	

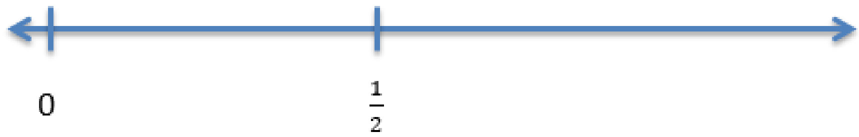
13. Fractions and Number Lines

Number lines are useful to help learners understand the relative sizes of fractions. Use examples such as the figures below.

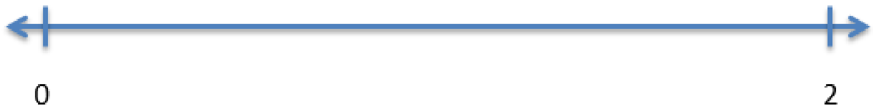
Mark $\frac{3}{4}$ on this number line:



Mark $\frac{3}{5}$ on this number line:

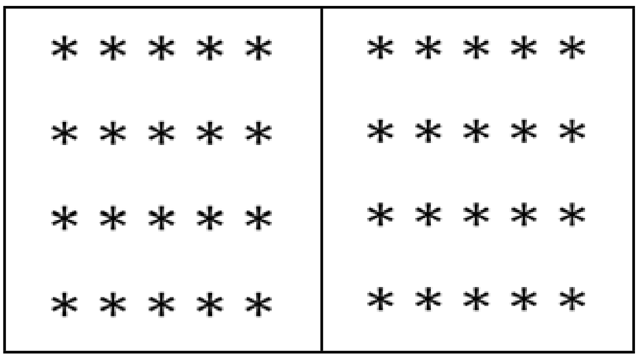


Mark 1; 0,7 ; $\frac{3}{4}$ on this number line:



14. Word problem

The rectangle of *'s below is 4/5 of the original rectangle of *'s. Draw or mark the original rectangle. The same rectangle of *'s below is now 5/4 of the original rectangle of *'s. Draw or mark the original rectangle.



This question lets learners think about what the unit is in each fraction. It underlines the importance of keeping the unit in mind when comparing fractions.

Section 3: Elements of Algebra

Introduction

Algebra is a key tipping point in the study of mathematics for many children (Mason, 2008). Before, mathematics makes sense to children, but algebra does not make sense to them anymore. They don't see a link between algebra and their daily life. More often, they experience a strong gap between the concrete work with numbers and operations and the abstract nature of algebra (Kainulainen et al., 2017). Therefore, preparing learners for algebra (algebraic thinking) should start in the early grades, not through using x and y, but by introducing the ideas behind algebra, such as identifying patterns. Algebraic thinking needs to be a logical and cohesive thread in the mathematics curriculum from pre-school to high school (The National Council of Teachers of Mathematics (NCTM), 2007). However, rushing students to represent patterns with letter symbols is counterproductive. Research on patterns suggests that it is generally more profitable for young students to explore for long periods of aspects of the generality in their patterns than to be exposed too quickly to the symbolic representation of this generality (Moss et al., 2006).

"Algebra is a key tipping point in the study of mathematics for many children. Before, mathematics makes sense to children, whereas algebra does not make sense to them anymore."

(Mason, 2008)

In the early grades, algebraic thinking comprises:

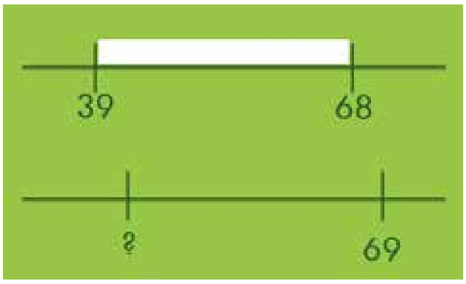
- sorting and classifying
- recognising and analysing patterns
- observing and representing relationships
- making generalisations
- analysing how things change

Young children are naturally curious about patterns and teachers can build on this curiosity. Children's work with patterns is an important developmental step on their journey towards algebraic thinking. For example, let learners generalise about things that are the same and different in patterns. As children explore and understand basic operations, they can look for patterns that help them learn procedures and facts such as exploring patterns in the multiplication tables. These are interesting to children and help them learn their multiplication facts and understand the relationship between facts.

In this section, we will discuss some key idea of algebraic thinking and suggest some activities that teachers can use to move from the concrete work with numbers and operations to the more abstract nature of algebra.

Relational Reasoning

Relational reasoning is about finding an unknown quantity without calculating, but by using the relationship between the numbers. For example, what should the missing number be on the second line to keep the size of the gap the same?



We keep the gap the same by …. .[increasing both numbers in the initial relationship by the same amount]. We have made a general statement. We can use the general statement to calculate missing numbers in problems without calculating. For example: 68 - 39 = 69 - __ = 70 - __ = ...

… = 271 - __ = __ - 140.

For example, take 375 + 99. It is easier to adjust the sum to make it 374 + 100. This is algebraic thinking. Research in UK with 11-year olds showed that only few learners used algebraic thinking, most worked out the sum (Mason, 2008).

Using **quasi-variables** helps learners to make bridges from existing arithmetic knowledge to algebraic thinking without having to rely on knowledge of algebraic symbols (Fujii & Stephens, 2008). Examples are open number sentences like 647 – 285 = [ ] – 300 or using a drawn cloud to represent the unknown (rather than x and y, which should only be introduced later).

- 64 + 14 = [ ] + 64
- 64 + [ ] = 18 + 64
- 64 + [ ] = 18 + 62

Extend to sums like: 86 + 57 = 143 -> 88 + 55 = same. Learners need to understand that this sum should give the same result.

Commutativity

Commutativity means that when we are adding, the order of the numbers does not matter. For young learners, the word is not important, but the idea. Using diagrams or blocks to introduce commutativity and show that the order in the addition does not matter. For children 99 + 3 is much easier than 3 + 99. Often children learn to put the first number in their head and count to the next number. We want children to recognize when it is easier to change the order of numbers in the sum.

Repeating and Growing Patterns

The main element to look out for in patterns is to expose learners to a variety of patterns (repeating and growing, arithmetic and geometric) and visualisations. Familiarize learners with the key elements for repeating and growing patterns:

Repeating patterns: The 'unit' that repeats

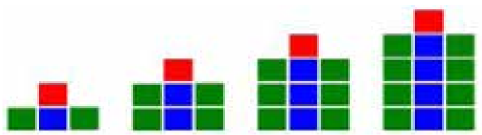
- How many elements in this unit?

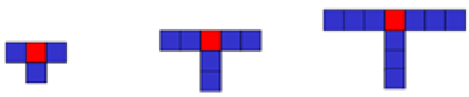
Growing patterns:

- How it starts
- How it grows

Learners need experience with growing patterns in both geometric and arithmetic (number) formats:

- ◊ 5, 6, __ , __ , 9 , __
- ◊ __, 9, 12, __ , 18, __
- ◊ 70, __, 50, __ , __ , 20
- ◊

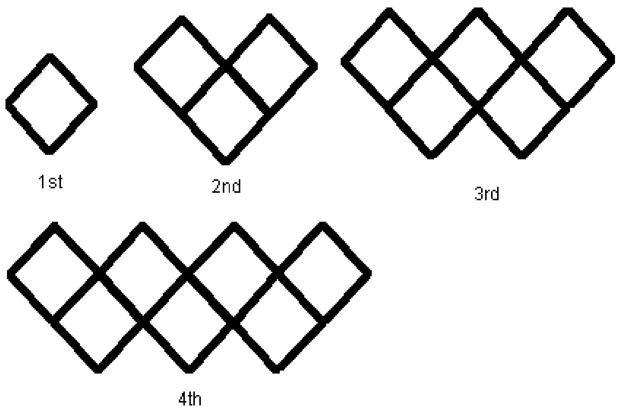




Some children have a limited understanding of patterns as only repeating. Children can extend patterns, but have trouble describing and generalizing patterns. Use many exercises where learners need to find elements far down the sequence (Moss et al., 2006). Mason (2008) suggests visualization and manipulation of geometric patterns as a step towards construction of the rule.

Gradually, teachers should move from word descriptions to numerical and algebraic descriptions. These allow to find out how the pattern will evolve without having to draw it.

For example, study the pattern, made of matchsticks, below.



How many squares and matchsticks would the:

- 10th picture have?
- 75th picture have?

Can you write **a general rule**? How many squares and matchsticks would the n-th picture have? Use the table below. Notice that with this question, learners gradually move to more abstract problem solving.

Picture	1	2	3	4	5
Squares	1	3	5	7	9
Matchsticks	4	10	16		

Activities

1. True or False (relational reasoning)

Let the learners find out, discuss and explain why:

- 37 + 56 = 56 + 37
- 37 + 56 = 38 + 59
- 37 + 56 = 38 + 57
- 37 + 56 – 56 = 37
- 458 + 347 – 347 = 458
- 56 – 38 = 56 – 37 – 1
- 56 – 38 = 56 – 39 + 1
- 56 – 38 = 56 – 36 – 2
- 3 x 5 = 3 x 4 + 5
- 3 x 5 = 3 x 4 + 3
- 64 ÷ 14 = 32 ÷ 28
- 64 ÷ 14 = 32 ÷ 7
- 42 ÷ 16 = 84 ÷ 32

Next, move to open number sentences (using quasi-variables, represented by open brackets):

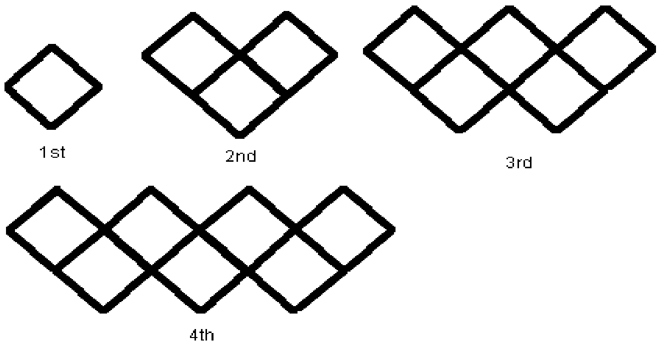
- 64 + 14 = [ ] + 64
- 64 + [ ] = 18 + 64
- 64 + [ ] = 18 + 62
- 647 – 285 = [ ] – 300
- 671 – 285 = 640 – [ ]
- [ ] – 285 = 640 – 285
- 3 x 5 = 5 x [ ]
- 3 x 5 = 3 x 4 + [ ]
- 3 x [ ] = 3 x 5 + 3

2. Repeating Patterns

What is the repeating unit? How many elements does the repeating unit contain? Create a repeating pattern with a repeating unit with 4 elements. Can you continue your repeating pattern? What would go in 84th position? The 407th position? Ask a partner to answer these questions for your pattern. Create a repeating pattern with a repeating unit with four elements using only '0' and '1'

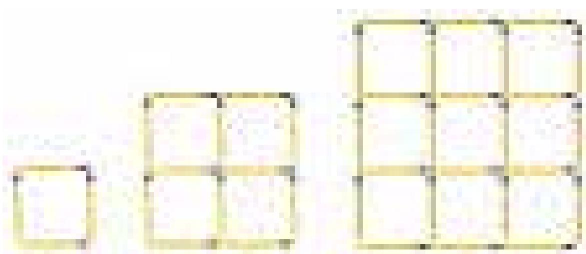
3. Growing Patterns

Make the matchstick pattern in the figure below.



- How many squares are there in the 5th and 6th patterns?
- How many matchsticks are there in the 5th and 6th patterns?
- How many squares/matchsticks are there in the 12th/13th/23rd /407th positions?
- How can we work out how many squares and matchsticks there will be (general rule):
    - in the 18th picture
    - In the 97th picture

Other growing patterns:

76, 73, 70, 67, 64, …	

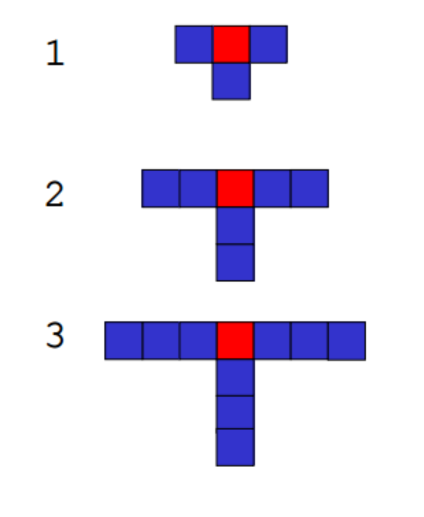
- Describe the pattern precisely in words? Can you describe how it grows?
- Now express the pattern with numbers.
- Work out the next few numbers.
- Work out the number in the 114th position.

Make a repeating pattern and then a growing pattern with your matchsticks.

Describe your pattern in words using the critical features mentioned above

Give your description to a partner on another table. Can they re-create your pattern?

Let learners analyse the pattern below. Is it a growing or a repeating pattern? Have them make a table (see below). How many blocks are there at the 6th position (Note: common mistake = 20, versus 19). How many blocks at the n^{th} position?

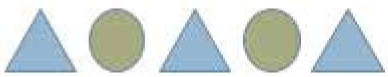


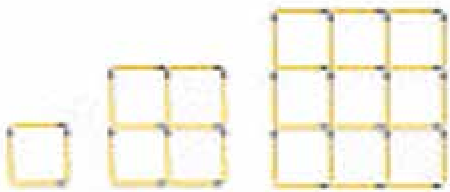
Position	Blocks
1	4
2	7
3	10
4	13
5	16

4. Distinguishing repeating and growing patterns

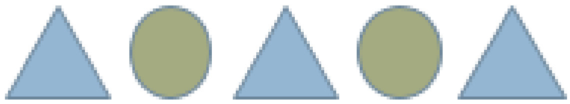
Can you describe these patterns? What kinds of questions can we ask about these patterns?

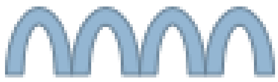
In what ways are these two patterns similar? In what ways are they different from each other?





Continue the pattern. Describe the pattern. In what ways are these four patterns similar to each other? How are they different from each other? What would be in the 12^{th} position in each pattern? The 13^{th} position? The 23^{rd} position? The 108th position?

1 

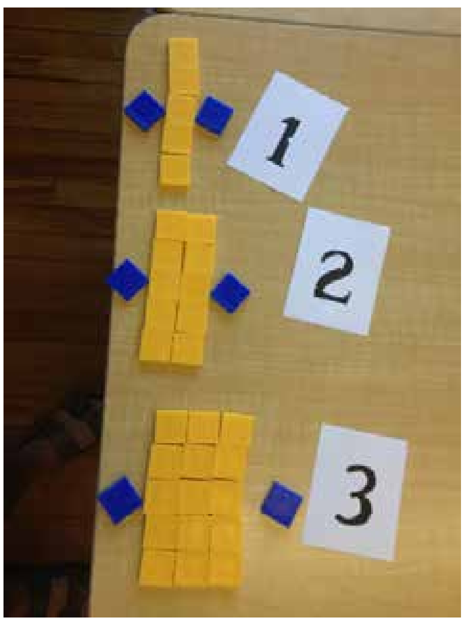
2 

3 X J J X J J X J

4 0 0 1 1 0 0 1 1

5. Using pattern cards or counters to let learners explore patterns

Use counters and number cards to let learners construct their own pattern. Next, you can let learners try and recognize each other's patterns. They must also be able to describe the pattern (repeating or growing, unit…).



Use group work and class discussion to construct understanding on patterns:

1. What makes some patterns easy for others to recognize the rule?
2. Think of one of the patterns around the room that might have been more difficult for you to figure out.

Sources:
https://buildingmathematicians.wordpress.com/2016/08/27/how-do-you-give-feedback/

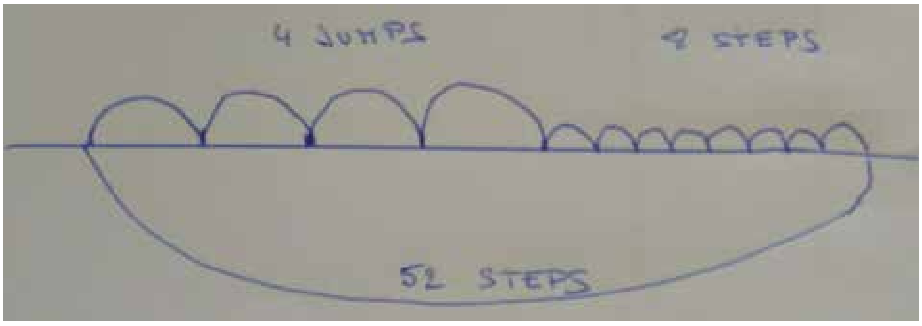
http://www.nelson.com/linearrelationships/From%20Patterns%20to%20Algebra%20Sampler%202012.pdf

6. Word problem: Frog activity

This problem introduces algebraic thinking, without already using symbolic language.

Francine the frog is a champion precision jumper. All her jumps are the same size (as are her steps). Francine makes 4 jumps and 8 steps. For her that is exactly the same as 52 steps. How many steps is a jump?

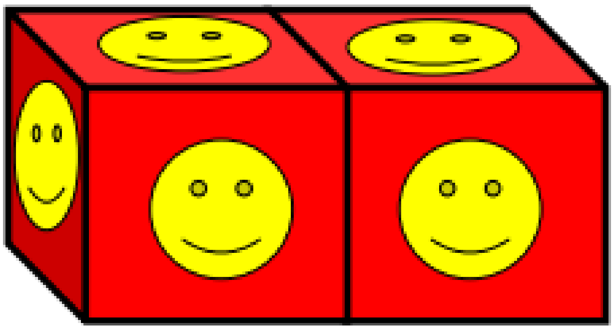
A good strategy to deal with this kind of word problems is "specialize, then generalize". First, explore the specific problem with drawings, tables etc. Then, use other numbers and try to find a general rule. You can find an example of a drawing below.



7. Cube stickers (Moss et al., 2006)

This is another word problem that introduces algebraic thinking.

A company makes coloured rods by joining cubes in a row and using a sticker machine to put "smiley" stickers on the rods. The machine places exactly 1 sticker on each exposed face of each cube. Every exposed face of each cube has to have a sticker. This rod of length 2 (2 cubes) would need 10 stickers.

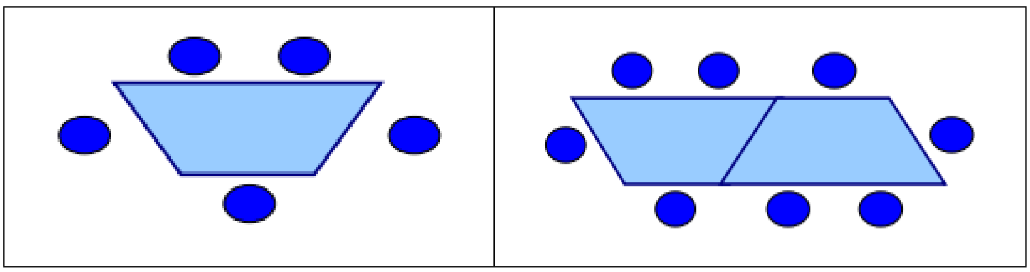


How many stickers would you need for:

- A rod of 3 cubes
- A rod of 4 cubes
- A rod of 10 cubes
- A rod of 22 cubes
- A rod of 56 cubes
- What is the general rule?

8. Trapezoid Tables (Moss et al., 2006)

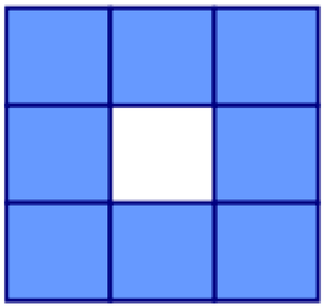
Nicolette decided she would place the chairs around each table so that 2 chairs will go on the long side of the trapezoid and one chair on every other side of the table. In this way, 5 students can sit around 1 table. Then, she found that she could join 2 tables like in the figure below, so that now 8 students can sit around 2 tables.



- How many students can sit around 3 tables joined this way?
- How many students can sit around 56 tables?
- What is the rule? How did you figure it out?

9. Perimeter Problem (Moss et al., 2006)

This is a 3x3 grid of squares with only the squares at the outside edge shaded. If you had a 5x5 grid of squares where only the outside edge of squares is shaded, how many squares would be shaded? If you had a grid of 100 number of squares, how many would be shaded? Is there a rule? How did you figure it out?

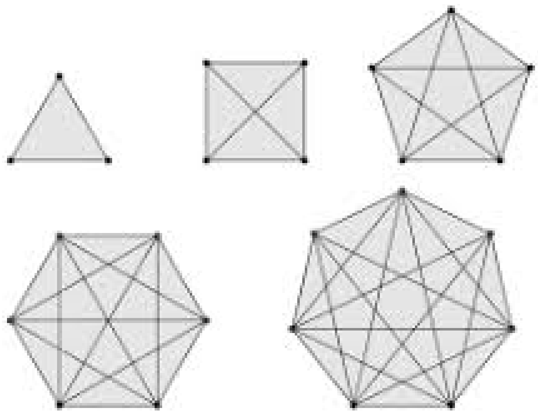


10. Handshake Problem (Moss et al., 2006)

Imagine that you are at a huge party. Everyone starts to shake hands with all the other people who are there. The problem can be represented by a table or by a drawing (see table and figure below).

- If 2 people shake hands, there is 1 handshake.
- If 3 people are in a group and they each shake hands with the other people in the group, there are 3 handshakes.
- How many handshakes if there are 4 people? 10 people? Can you use a rule to help you figure this out?

People	Handshakes
2	1
3	3
4	6
5	10
6	15
7	21



11. Linking Patterns to Generalization

An important step to move from arithmetic to algebra is to recognize and describe patterns. You can use exercises such as this one to let learners generalize patterns.

- Choose an even number
- Choose another even number
- Add them together
- What kind of number do we get?
- Choose another pair of even numbers
- Is the result the same kind of number?
- Will the result always be the same kind of number?
- Use a diagram or a word explanation to show why your result is true.
- What about even + odd?
- What about odd + even?
- What about odd + odd?
- Try this:
  - 1 + 3
  - 1 + 3 + 5
  - 1 + 3 + 5 + 7
  - …
  - What can you say about the results?

12. Word problems of type "Think-of-a-Number (TOAN)

If you know the sum and the difference of two numbers, can you figure out what the two numbers are? Example: A + B = 13 and A – B = 5. List all sums and differences and look for pattern. Learners should eventually find out that A can be found by taking (sum + difference)/2.

A variation of this

- Example: TOAN, add 5, double, add 2, half the answer and subtract your original number. The result is always six!

- Play the game a few times with different numbers.
- Try and find the explanation why the result is always six. Come to a generalized statement (using A and B).

You can challenge learners to make their own TOAN activity.

13. Word problem: Buying T-Shirts

Concord Trading sells T-shirts for 3000 Frw each, but adds a delivery charge of 5000 Frw regardless of how many T-shirts you order. True Sports sells the same T-shirt for 4000 Frw each without any delivery charge. Better still, for every order, True Sports gives a discount of 2000 Frw on the entire bill.

What is the cost of buying 5 T-shirts from each store? Of buying 10 T-shirts? For which number of T-shirts will the price be the same in both shops?

Source: https://elsdunbar.wordpress.com/2016/05/27/learning-from-a-5th-grade-math-team/

Section 4: Probability and Statistics

A key learning outcome in probability and statistics is to make learners familiar with the concept of probability. Through a variety of daily life situations and using concrete materials, learners explore questions like:

- How likely is something going to happen?
- What are the chances of an event happening?

Examples of concrete materials and situations are:

- counters in different colours: "how likely am I going to pick a blue counter?"
- dice: "what is the chance of throwing a 5?"
- spinners: see below
- learners themselves:
  - how likely is it that a learner's birthday falls in November?
  - how likely to pick a girl if teacher picks a name at random?

Below we provide some ideas on how you can use **self-made spinners** in your lessons on probability.

1. Using spinners

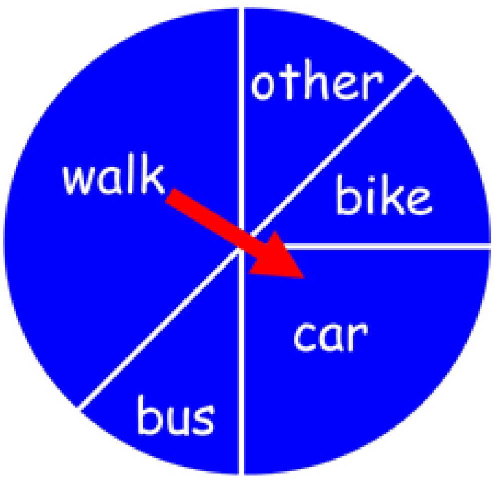
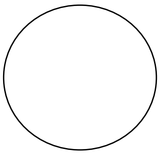
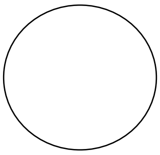
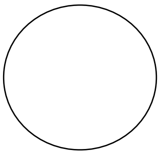
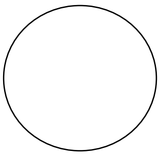
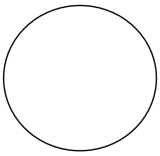
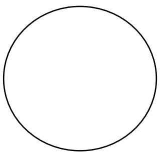
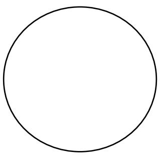


Figure 7: Example of a spinner

Let learners draw spinners based on guidelines that you provide. Probabilities in different mathematical notations are introduced and practiced, as well as terminology such as likely, unlikely and certain.

Table 3: Colour in the spinners to show the different probabilities.

	50:50 chance of red or white
	$\frac{3}{4}$ chance that you will get blue
	More likely than you will get red than green and less likely that you will yellow than red
	Certain that you will get a yellow
	Unlikely that you will get red Likely that you will get yellow Not impossible to get green
	Where it is impossible to get red but likely to get white.
	3 in 8 chance that you will get red 2 in 8 chance that you will get blue Impossible to get yellow

In a subsequent activity, learners can design their own spinners and explain the probability of landing on different colours, using a worksheet like the one below.

Table 4: Colour in the spinners to show the probabilities that you define

2. Investigation activity to introduce probability

To introduce the concept of probability, you can use the following investigation activity with the learners.

1. Let learners in pairs toss a coin in the air for a total of 30 times.

- Let them predict how many times they will have head or tail.
- Every time the coin lands they record whether they get a 'head' or a 'tail'.
- They write H for 'head' and T for 'tail' in a table.
- How many times did you get a 'head' (H)? ________________
- How many times did you get a 'tail' (T)? ________________

2. Secondly, learners roll a dice for a total of 30 times.

- Let them predict how many times they will throw a 1, 2…
- Every time they roll, they record the score on the dice and write it in a table.

Number of 1s: __________ Number of 4s: ________

Number of 2s: __________ Number of 5s: ________

Number of 3s: __________ Number of 6s: ________

- Further questions you can ask for discussion:
- If we rolled 2 coins what possible outcomes could we get?
- If we rolled more than 1 dice what possible outcomes could we get?
- What would our chances of getting 2 heads or a 6 be like then?

APPENDIX 2: SELF-EVALUATION FOR PRIMARY MATHEMATICS TEACHERS

How confident are you to apply appropriately following techniques for mathematics teaching?

Techniques for mathematics teaching	Not confident	A bit confident	Quite confident	Very confident
Questioning				
Use open questions to challenge pupils and encourage them to think				
Use voting to involve all learners when asking questions				
Stimulate interactions between learners and not only between the teacher and learners when asking questions				
Let learners formulate mathematical questions themselves				
Mathematics Conversations				
Use techniques that stimulate learners to express their mathematical ideas				
Help learners to master key mathematical vocabulary,				

Developing problem solving skills				
Use activities that stimulate the development of problem-solving skills with learners				
Use word problems that stimulate learner's thinking and understanding of mathematical concepts.				
Teach learners to formulate their own mathematical problems				
Learner errors and misconceptions				
Be familiar with common mathematical misconceptions that learners have				
Use techniques to expose and change learner misconceptions about mathematics				
Connecting concrete, pictorial and abstract representations of mathematical concepts				
Introduce a mathematical concept with concrete materials or experiences, and gradually move to pictorial and abstract representations of the concept.				
Use low-cost materials to teach and learn mathematics				
Games				
Use games to increase understanding about mathematical concepts				

Use games to practice basic mathematical skills such as number sense and operations.				
Gender and Inclusiveness in mathematics				
Address gender stereotypes about mathematics				
Make sure that all learners have equal opportunities to achieve the learning outcomes in mathematics				
Use differentiation to make learning mathematics more inclusive				
Assessment				
Use formative assessment to inform yourself and learners about their learning.				
Use the results from formative assessment to change your teaching.				

Based on your self-evaluation above, formulate 3 priorities for yourself in this CPD Programme.

1.

2.

3.

APPENDIX 3: CONCEPT CARTOONS FOR PRIMARY MATHEMATICS

Figure 8: Concept Cartoon "Multiplication"

Figure 9: Concept Cartoon "City Temperatures" (Millgate House Education, 2008, adapted by VVOB Rwanda)

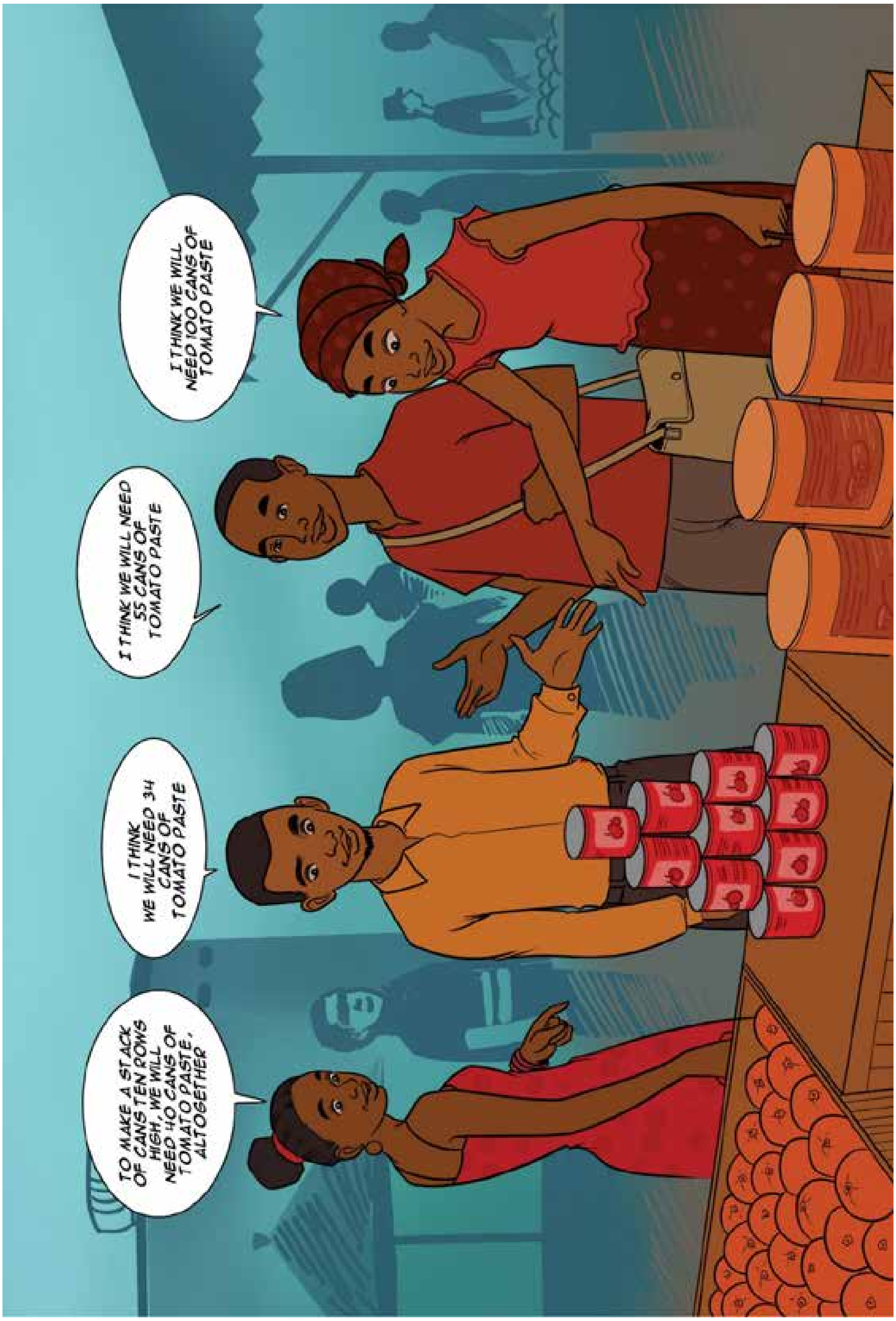


Figure 10: Concept Cartoon "Tomato Cans" (Millgate House Education, 2008, adapted by VVOB Rwanda)

Figure 11: Concept Cartoon "Tessellation" (Millgate House Education, 2008, adapted by VVOB Rwanda)

Figure 12: Concept Cartoon "Umuvure" (Millgate House Education, 2008, adapted by VVOB Rwanda)

Figure 13: Concept Cartoon "boxes" (Millgate House Education, 2008, adapted by VVOB Rwanda)

Figure 14: Concept Cartoon "Circular Lawn" (Millgate House Education, 2008, adapted by VVOB Rwanda)

NEWS WEBSITES	NUMBER OF READERS
IGIHE	15
INYARWANDA	45
RWANDA TODAY	10
THE NEW TIMES	20
TOTAL	90

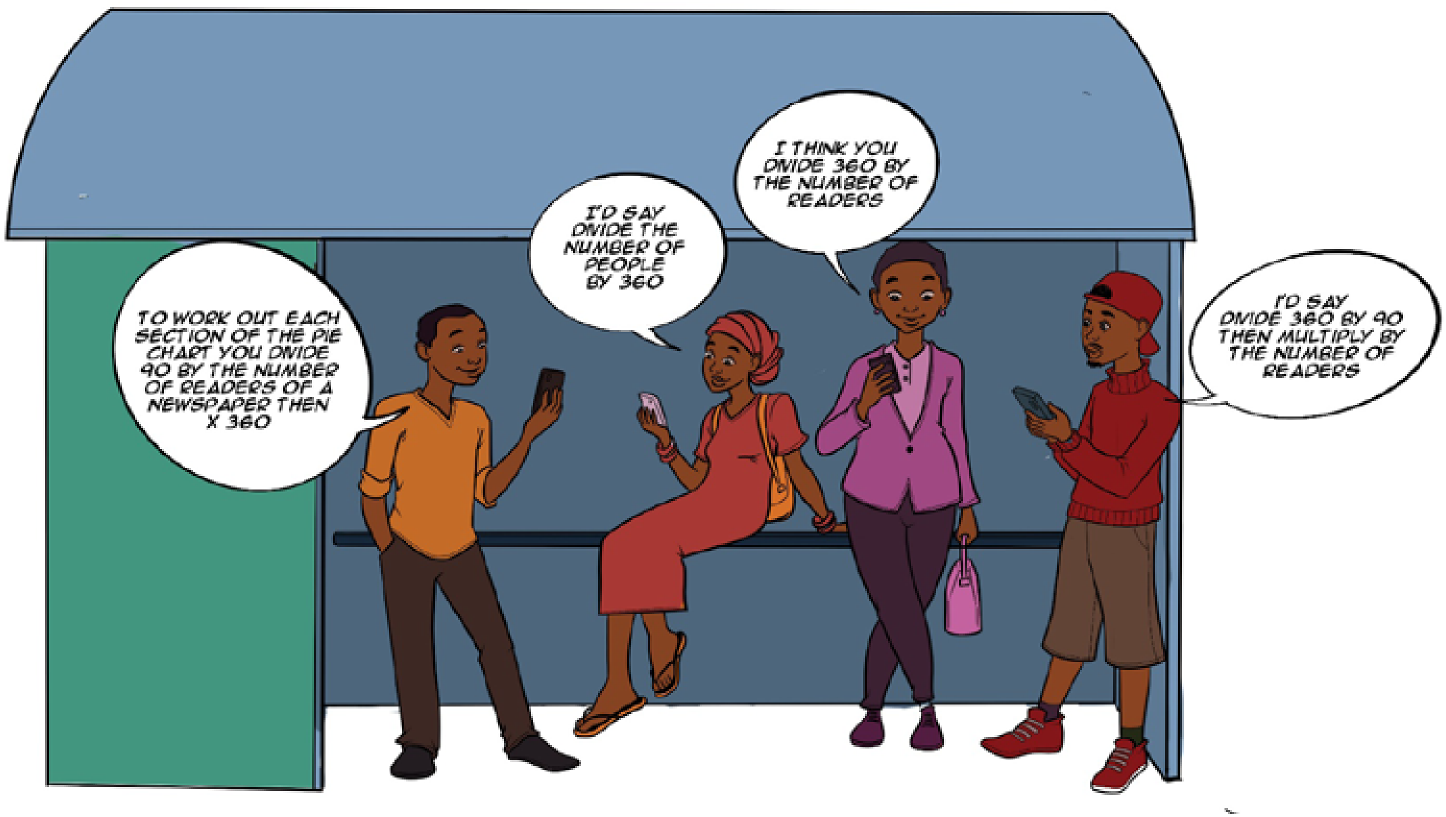


Figure 15: Concept Cartoon "Newspapers" (Millgate House Education, 2008, adapted by VVOB Rwanda)

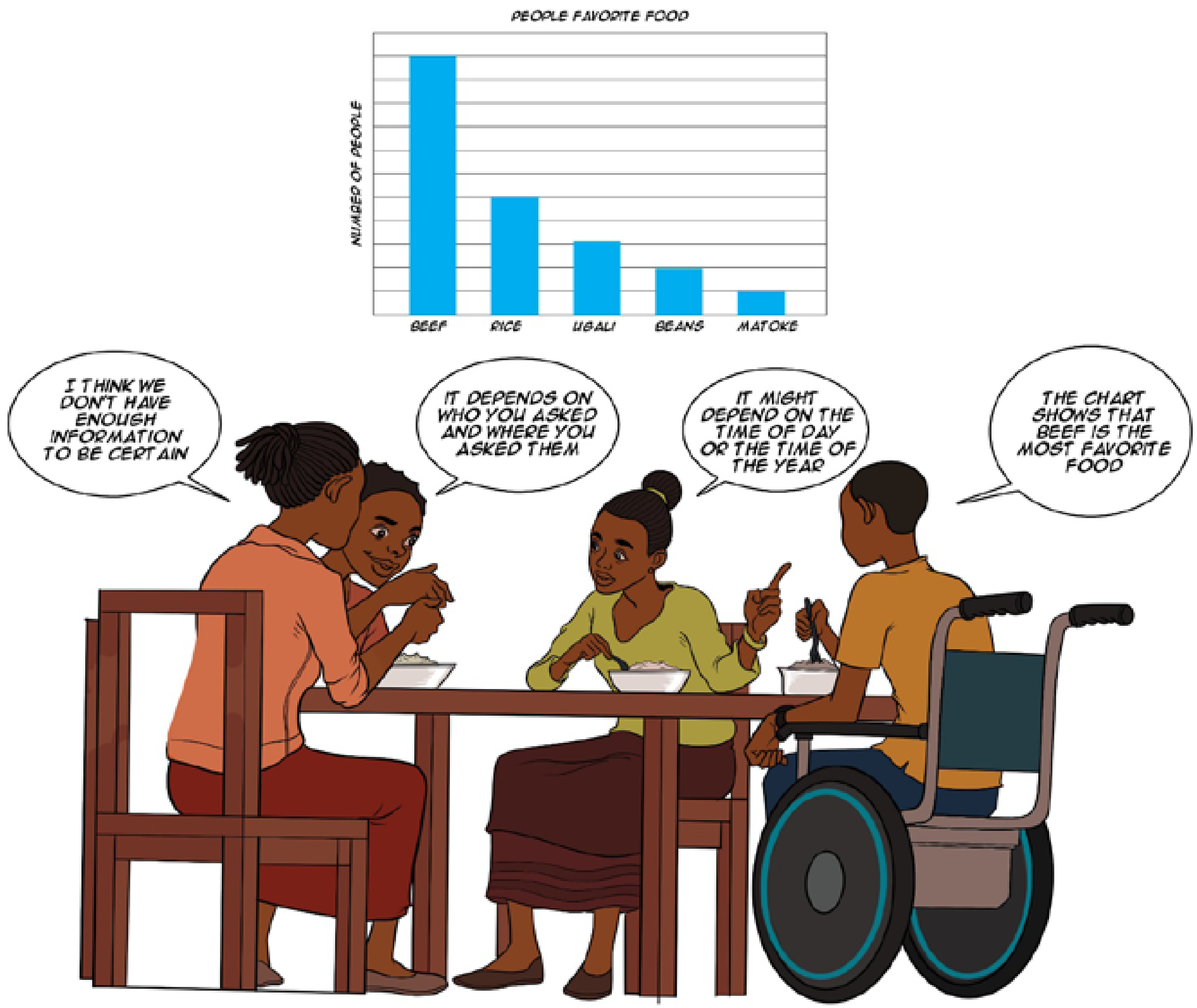


Figure 16: Concept Cartoon "Favourite Food" (Millgate House Education, 2008, adapted by VVOB Rwanda)

Figure 17: Concept Cartoon "Lottery" (Millgate House Education, 2008, adapted by VVOB Rwanda)

Figure 18: Concept Cartoon "Playing Cards" (Millgate House Education, 2008, adapted by VVOB Rwanda)

Discussion about the Concept Cartoons

1. Multiplication

Whenever a number is multiplied by a fraction, the answer isn't always small. Lots of numbers get smaller when multiplied by a fraction for instance, 10. However, there are some exceptions, depending on which numbers you multiply. These include multiplying by an improper fraction such as 2 or . In these examples, the answer isn't smaller. Multiplying by a mixed number will produce a bigger number, for example 10 . What about multiplying two fractions together, such as ? What about negative numbers? For a teacher to address the misconception, he/she may give several examples for learners to explore and identify the differences. (i.e multiply by improper fractions, mixed fractions, etc).

2. City Temperatures

A misconception in the cartoon above is to ignore the integer signs and look for two numbers which make a difference of 14 such as Kigali and Beijing. Another common mistake is to ignore the integer signs and look for two numbers with a total of 14 such as Moscow and Beijing. However, integer signs should be considered (For example, the difference between Kigali and Beijing is ; the difference between Moscow and Beijing is . In the cartoon above, the two cities with a difference of 14 degrees are London and Moscow . To address the misconception, learners should not forget integer signs.

3. Piles of Tomato Cans

The cartoon is about operations on consecutive numbers. Some possible miscocnceptions/ mistakes would be to multiply the cans in the stack by 10 to make 100 cans because there are ten layers or rows. Another mistake is to multiply the four cans on the bottow row in the picture by 10. However, it is not possible to work out the number of cans by a simple multiplication sum. For instance, one way to solve the puzzle is to draw the cans row by row on a mini-whiteboard. In a ten row stack, there will be 1+2+3+4+5+6+7+8+9+10= **55 cans**. A quick way to work out this is with N the quantity of consecutive numbers or rows. For the stack of cans we get will be needed to make a stack of cans ten rows high.

4. Tessellation

The cartoon above introduces different geometrical shapes. For learners to solve the puzzle above, give them different shapes and ask them which ones will tessellate. Tessellation is a repeating pattern made of identical flat shapes that cover a plane completely without overlapping. They fit perfectly together without leaving any gaps. A misconception is that many people believe that only regular pentagons will tesselate. However, some irregular pentagons can tessellate as well. Only three regular polygons do tessellate: triangles, squares and hexagons.

5. Umuvure

The cartoon above discusses multiplication and division of fractions. Try with simple examples such as ½ of 4 and write this in different orders such as 2 : 4 × 1, or 1 × 4: 2 or 4 : 2 × 1 or 1 : 2 × 4 and so on. Which order gives the correct ? For the given cartoon, it is possible to solve 3/4 of 80 in different ways. It could be solved as (3 × 80) : 4 or (80 : 4) × 3 or 0.75 × 80. All of these give the same answers. Three quarters of 80 is 60 banana. It is important to understand the word " of" as meaning of multiply, so ¾ of the banana means multiply the number of bananas by ¾. Brackets help to organize the numbers to identify which part of the sum should be worked out first. Does the same order work for improper fractions, such as 12/5 of 65?

6. Boxes

To sort the puzzle out, the volume of the biscuits box is . The volume of the parking crate is . So, the number of boxes that should fit in the crate is 216 000: 1080 = 200 boxes. However, it is not as simple as this. The shape of the boxes means that not all the space in the crate can be used. The boxes don't stack exactly to use all the space. The most that will fit in the crate is 198 boxes.

7. Circular Lawn

To solve question in the cartoon above, find the area of a circle which is ∏ × radius squared. ∏ being 3.14. The area of the circular lawn will be 3.14 × 16.5 × 16.5 = 854.865 sqm. Each box of the galloon (or container) treats 854.865 sqm : 100 sqm= 8.54 galloons. This more than 8 boxes, so 9 boxes must be needed.

8. Newspapers

This cartoon discusses data representation and analysis using a pie chart. A pie chart looks like a pie cut into slices. In the cartoon above, each slice will represent the number of readers who read a particular newspaper. If most of the people read one of the papers, its slice will be very big. To get the size of the slices we need to work out how much of the circle to use for each one. There are 90 reader altogether. They will fill the whole circle. To turn through a complete circle we rotate 360^0 So to work out each reader's slice, we divide the rotation of a complete circle by the total number of readers: $360^0 : 90 = 4^0$ If 15 people read Igihe, then this is $15 \times 4^0 = 60^0$ If 45 people read Inyarwanda, then this is $45 \times 4^0 = 180^0$. If 10 people read Kigali Today, then this is $10 \times 4^0 = 40$ 0. If 20 people read the New Times, then this is $20 \times 4^0 = 80^0$

9. Favourite Food

The cartoon above discusses and interprets data using a bar chart. Which food seems to be most popular? Which one seems to be the least popular? While analysing, a misconception which might arise is to conclude that beef is the most favourite food. However, we don't have enough information to be certain. People's favourite food might depend on who was surveyed (e.g if people who were surveyed stay in the countryside and do not always afford money to buy beef then they may be more likely to prefer beef over anything else). It could depend where the people who were surveyed were and what time of the year (e.g if they were in a butchery or it was during festive seasons). Therefore, all the answers could be correct.

10. Lottery

For real world problems that involve ratios, the formula is to add ratios e.g. 2+3=5 to find out how much money will be needed altogether. Find the value of one share 60,000Rfw: 5= 12,000Rfw. Then find amount for each person. David will get 3 lots which are worth 12,000Rfw × 3= 36,000Rfw while his brother will get 2 lots which are worth 12,000Rfw × 2 = 24,000Rfw. This method would also work for ratios involving 3 people.

11. Playing Cards

The cartoon above discusses the probability using playing cards. You can start this by making a small pack of 20 cards consisting of the Ace, King, Queen, Jack and the ten of all suits. What is the probability of selecting an Ace from this pack? What about the Queen of Spades? A red card? Any King or Queen? Any card except a 10? What is the probability of selecting each of these cards from the full pack of cards? To work out these questions, the probability of drawing a Spade from a deck of cards is 13 in 53 as there are 13 Spades in a pack of 52 playing cards. This can be simplified to ¼. If the Ace of Spades has already been drawn out, there are only 12 spades left in the next 51 cards. So the probability of the next cards being a spade is 12/52 . If the Ace of Spades is put back in the pack and the pack is shuffled, the probability of the next card being a Spade is now 13/52 or ¼.

APPENDIX 4: SAMPLE LESSON PLAN OF PRIMARY MATHEMATICS LESSON

IMBATA Y'ISOMO RY'IMIBARE – UMWAKA WA KABIRI (REB, Play-Based Learning Teacher guide for primary school, 2016 in progress)

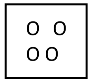
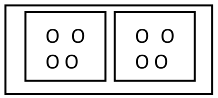
Izina ry'ishuri: GS AKANZU **Amazina y'umwarimu:** MUSABYIMANA Daniel

Igihembwe:	**Itariki:**	**Inyigisho**	**Umwaka wa**	**Umutwe wa**	**Isomo rya**	**Igihe isomo rimara**	**Umubare w'abanyeshuri**
Cya 1	12 Gashyantare 2016	Imibare	Kabiri	2	5/24	Iminota 40	46
Abafite ibyo bagenerwa byihariye mu myigire no mu myigishirize n'umubare wabo				Umwana umwe ufite ubumuga bw'ingingo (Agendera ku mbago imwe) aricara hafi y'ikibaho.			
Umutwe		Imibare kuva kuri 0 kugera 500					
Ubushobozi bw'ingenzi bugamijwe		▪ Kubara, gusoma, kwandika, gutondeka, kugereranya, guteranya gukuba no kugabanya neza imibare ishyitse kuva kuri 0 kugera kuri 500					
Isomo		Mara ya 4					
Intego ngenamukoro		Hifashishijwe udupapuro twanditseho imibare 1 kugeza 10 n'agakino kitwa "Dukine dukuba na 4" buri munyeshuri araba ashobora gukuba neza umubare afite na kane kandi akabivuga mu ijwi riranguruye mu minota 5					

Imiterere y'aho isomo ribera	Mu ishuri, Abanyeshuri baricara mu ishusho ya U. Hagati y'ikibaho n'abanyeshuri harasigara nibura metero 3
Imfashanyigisho	▪ udukarito 10 turimo udufuka, buri gafuka karimo amabuye ane ▪ Udupapuro twanditseho imibare kuva kuri 1 kugera ku 10, ikibaho, ingwa, ikarito irimo ibikoresho 40. (amabuye, udufuniko tw'amacupa)
Imyandiko n'ibitabo byifashishijwe	Integanyanyigisho y'imibare icyiciro cya mbere cy'amashuri abanza pge 16, Numeracy learning through play

Ibice by'isomo + igihe	**Gusobanura muri make igikorwa umwarimu n'umunyeshuri basabwa gukora**		**Ubushobozi n'ingingo nsanganyamasomo (andika ubushobozi + igisobanuro kigufi kigaragaza**
	Hifashishijwe udufuka turimo utubuye tunetune. Mu matsinda mato, abanyeshuri bashaka umubare w'utubuye twose turi mu gakarito. Bafashijwe na mwarimu, batahura ko iyo uteranyije ibintu bingana ku buryo bwisubiramo bingana no kubikuba izo nshuro.		
	Ibikorwa by'umwarimu	**Ibikorwa by'umunyeshuri**	
Intangiriro: Iminota 5	➢ Gusaba abanyeshuri kuvuga mara ya kabiri mu njyana.	➢ Kuvuga mu njyana mara ya 2	
Isomo nyirizina: **Iminota 25**	➢ Gufasha abanyeshuri gukora amatsinda atandatu ➢ Gutanga amabwiriza y'umukino	➢ Gukora amatsinda bakurikije amabwiriza y'umwarimu.	**Uburezi budaheza:** umwarimu yita ku mwana ufite ubumuga bw'ingingo

a. igikorwa k'ivumbura	➢ Kwereka abanyeshuri ibikoresho bitandukanye biri bwifashishwe ▪ Dufite udukarito 10. ▪ Buri gakarito karimo udufuka turimo amabuye 4 muri kamwe ▪ Buri tsinda rirohereza umuntu umwe uza gutombora ▪ Muratanguranwa kubwira umubare w'amabuye yose ari mu gakarito mwatomboye ▪ Utsinda ni uravuga umubare nyawo kandi akabyandika ku buryo bworoshye kubisoma.	➢ Mu matsinda abanyeshuri barakina agakino bubahiriza amabwiriza yatanzwe ku buryo bukurikira: ▪ Buri tsinda rirohereza umunyeshuri umwe gutombora agakarito kamwe ▪ Abagize itsinda barabara byihuse utubuye twose turi muri buri gafuka, babashe kubona utubuye twose turi muri buri gakarito ▪ Buri tsinda rirandika umubare w'utubuye babonye bagaragaza n'uko babigezeho.	kandi akaba hamwe n'abandi amwicaza imbere. **Ubufatanye, imibanire ikwiye n'abandi n'ubumenyi ngiro mu buzima bwa buri munsi:** mu gihe abanyeshuri bakorera hamwe mu matsinda bashaka ibisubizo
b. Isesengura	➢ Kubaza ibibazo birebana n'uko bageze ku gisubizo ○ Ni iki cyabashimishije muri aka gakino ? ○ Ni gute mwageze ku gisubizo?	▪ Buri tsinda riragenda risobanura uko bageze ku gisubizo	**Ubushishozi no gushakira ibibazo ibisubizo:** Mu gihe abanyeshuri basobanura inzira baciyemo kugira ngo bagere ku gisubizo cyabo

	○ N'iki cyabagoye muri aka gakino ➢ Kureba abakoresheje guteranya no kubabaza inshuro bateranyije no kubabaza ikimenyetso gikoreshwa mu mibare iyo ushaka kugaragaza inshuro wakoze ikintu. ➢ Kureba kubakoresheje uburyo bwo gukuba akabereka ko  4x1 = 4  4+4 = 4x2 = 8 ➢ 4+4+4 = 4x3 = 12 ➢ 4+4+4+4 = 4x4 = 16 ➢ 4+4+4+4+4 = 4x5 = 20 .... gukomeza gutya kugeza ku gikubo '40'	▪ Buri munyeshuri aravuga ibyagiye bimukomerera ▪ Buri munyeshuri aregenda yandika mu ikayi isano riri hagati yo gukuba no guteranya ➢ 4 = 4x1 = 4 ➢ 4+4 = 4x2 = 8 ➢ 4+4+4 = 4x3 = 12 ➢ 4+4+4+4 = 4x4 = 16 ➢ 4+4+4+4+4 = 4x5 = 20	**Ubufatanye, imibanire ikwiye n'abandi:** mu gihe abanyeshuri bakorera hamwe mu matsinda bashaka ibisubizo by'ibibazo bahawe.

c) Ikomatanya	Kwandika mara ya kane kuva kuri rimwe kugeza ku icumi 4x1=4 4x2= 8 … 4x10=40 Gukoresha umwitozo wo gufata mu mutwe mara ya kane	Abanyeshuri barasubiramo inshuro nyinshi mara ya kane kugira ngo bayifate mu mutwe.	
Umusozo w'isomo: (Isuzuma) **Iminota 10**	➢ Gutanga amabwiriza y'agakino k'isuzuma ko gukina bakuba na kane anerekana ibikoresho (udupapuro) biri bwifashishwe ➢ Gukurikirana imigendekere y'agakino mu matsinda atandukanye	Mu matsinda ya babiribabiri, abanyeshuri barakina umukino wo gukuba; ▪ umunyeshuri wa mbere araatombora agapapuro kanditseho umubare (1 kugera 10). ▪ Ahisha agapapuro mu kiganza kimwe ▪ asaba mugenzi we gufindura ikiganza gihishemo agapapuro	**Gusabana mu ndimi zemewe gukoreshwa mu gihugu :** buzagaragarira mu kuganira hagati y'abanyeshuri na mwarimu ndetse n'abanyeshuri ubwabo.

	➢ Gusaba amatsinda amwe kujya imbere no gukina agakino abandi bose bakurikiye ➢ kuyobora ikiganiro gisoza ▪ Uyu munsi twize iki? ▪ Niba ushaka kugira amakaye abiri muri buri somo mu masomo ane akurikira: Ikinyarwanda, Imibare Icyongereza n’imbonezamubano, ubwo mu	▪ mugenzi we afindura ikiganza kirimo agapapuro ▪ iyo agasanzemo, asoma umubare urimo ▪ akuba uwo mubare na kane maze akavuga igisubizo ▪ Mugenzi we nawe ahita atombora ▪ umukino ukomeza gutyo mu matsinda ya babiribabiri. ➢ Gusubiza utubazo tugaragara mu kiganiro bagirana na mwarimu. ○ Uyu munsi twize mara ya kane… ○ Waba ukeneye amakaye 8	

	by'ukuri amakaye yose ukeneye ni angahe? ▪ Mu rugo muri abana bane. Mama agusabye kugurira buri wese bombo enye, Ubwo uzagura bombo zingahe kuri butike?	○ Uzagura bombo 16	
Kwisuzuma (umurezi)	Muir iri somo abanyeshuri bishimiye umukino bize kandi bize gukuba / mara ya kane binyuze mu mukino. Bavumbuye isono riri hagati yo gukuba no guteranya		

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